Concepts of System Stability
Higher-Order System response

Consider the closed loop transfer function

\[
\frac{C(s)}{R(s)} = G_c(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{N_c(s)}{D_c(s)}
\]

\[
= \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0}
\]

with \( n > m \)

The system is said to be a higher-order system for \( n > 2 \).
There will be \( n \) poles of \( G_c(s) \), or \( n \) roots of the Characteristic Equation

\[
D_c(s) = 0 \quad \text{or} \quad 1 + G(s)H(s) = 0.
\]

\( G_c(s) \) can also be written as

\[
G_c(s) = \frac{C(s)}{R(s)}
\]

\[
= \frac{K(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_q)(s^2 + 2\zeta_1\omega_1 s + \omega_1^2)(s^2 + 2\zeta_2\omega_2 s + \omega_2^2) \cdots (s^2 + 2\zeta_r\omega_r s + \omega_r^2)}
\]

where \( q + 2r = n \)

\[
= \sum_{j=1}^{q} \frac{a_j}{s + p_j} + \sum_{k=1}^{r} \frac{b_k (s + \zeta \omega_k) + c_k \omega_k \sqrt{1 - \zeta^2_k}}{s^2 + 2\zeta_k \omega_k s + \omega_k^2}
\]

using partial fraction expansion
Higher-Order System response

For a unit step input, we can re-write \( C(s) \) in terms of partial fractions as

\[
C(s) = \frac{a}{s} + \sum_{j=1}^{q} \frac{a_j}{s + p_j} + \sum_{k=1}^{r} \frac{b_k(s + \zeta \omega_k) + c_k \omega_k \sqrt{1 - \zeta^2_k}}{s^2 + 2\zeta_k \omega_k + \omega_k^2}
\]

in which we assume that all the poles are distinct, i.e. not repeated.

The time response will then be, by using the Inverse Laplace Transform

\[
c(t) = a + \sum a_j e^{-p_j t} + \sum_{k=1}^{r} \frac{e^{-\zeta_k \omega_k t}}{\sqrt{1 - \zeta^2_k}} \sin(\sqrt{1 - \zeta^2_k} \omega_k t + \phi_k)
\]

where \( \phi_k = \tan^{-1}\left(\frac{\sqrt{1 - \zeta^2_k}}{\zeta_k}\right) \)

For a stable response, the poles must all have negative real parts.

The response of a stable higher-order system thus comprises a sum of a number of decaying exponential curves and decaying damped sinusoidal curves together with a term dependent upon the input.
Higher-Order System response
Plot of poles on the s-plane

Real Poles and their effect on the response

\[ G(s) = \frac{C(s)}{R(s)} = \frac{K}{(s + p_1)(s + p_2)} \]

Each real pole will contribute a term \( a_j e^{-p_jt} \) into the response.

The more negative the pole, or the farther away to the left from the Imaginary axis it is, the more rapidly the exponential term decays to zero.

In general, if two poles are such that \( |p_1| > 5|p_2| \), then the response caused by \( p_2 \) is dominant and that for \( p_1 \) can be neglected without loss of accuracy.
Higher-Order System response
Plot of poles on the s-plane

Complex conjugate poles and their effect on the response

Each complex pair contributes a decaying damped sinusoidal term to the response.

\[ G(s) = \frac{C(s)}{R(s)} = \frac{K}{(s^2 + 2\zeta \omega_n s + \omega_n^2)} \]

\[ p_{1,2} = -\zeta \omega_k \pm j\sqrt{(\zeta^2 - 1)} \]

Length = \[ \sqrt{(-\zeta \omega_k)^2 + (1 - \zeta^2)\omega_k^2} = \omega_k \]

The more negative the real part \( -\zeta \omega_k \), or the farther away to the left the poles are from the Imaginary axis, the more rapidly the term decays to zero.

The angle \( \phi_k \) the poles make with the Real Axis determines the damping ratio, the greater the angle, the less the damping ratio.

\[ \phi_k = \tan^{-1} \left( \frac{\sqrt{1 - \zeta_k^2}}{\zeta_k} \right) \]

Complex conjugate poles on the s-Plane.

Lines of constant \( \zeta \)
Some typical responses

Stable systems

Overdamped

Underdamped

\[
G(s) = \frac{9}{s^2 + 9s + 9}
\]

\[
R(s) = \frac{1}{s}
\]

\[
C(s)
\]

\[
c(t) = 1 + 0.171e^{-7.854t} - 1.171e^{-1.146t}
\]

\[
c(t) = 1 - e^{-t}(\cos(\sqrt{8}t) + \frac{\sqrt{8}}{8} \sin(\sqrt{8}t))
\]

\[
= 1 - 1.06e^{-t} \cos(\sqrt{8}t - 19.47°)
\]
Some typical responses

Stable systems

(d) \( R(s) = \frac{1}{s} \):
\[
G(s) = \frac{9}{s^2 + 9}
\]
Undamped

(e) \( R(s) = \frac{1}{s} \):
\[
G(s) = \frac{9}{s^2 + 6s + 9}
\]
Critically damped
Some typical responses

An Unstable systems

Unstable system's closed-loop poles (not to scale)
Complex conjugate poles and their effect on the response

The relative dominance of closed-loop poles is determined by how far they are from the Imaginary Axis, assuming that there are no zeros nearby. (Zeros affect the relative magnitude of the constant terms associated with the poles, the closer they are the more the effect.)

Usually the response will be adjusted such that one pair of complex conjugate poles will be closer to the Imaginary Axis relative to all the other poles and the response caused by this pair dominates the overall response. This pair is called the dominant closed-loop poles.
Effect of closed-loop poles on response

OCTAVE Program

```octave
G_1(s) = \frac{1}{s + 1}

G_2(s) = \frac{10}{s + 10} = \frac{1}{0.1s + 1}

G_3(s) = \frac{10}{(s + 1)(s + 10)} = \frac{1}{(s + 1)(0.1s + 1)}
```

```octave
hold on
g1=zp([],[-1],1);
step(g1)
hold on
g2=zp([],[-10],10);
Step(g2)
hold on
g3=zp([],[-1 -10],10);
step(g3)
```
OCTAVE Program

\[ g_4 = \text{zp}([-4], [-1 -10], 10/4); \]
\[ \text{step}(g_4) \]
\[ \text{hold on} \]
\[ g_5 = \text{zp}([-1.1], [-1 -10], 10/1.1); \]
\[ \text{Step}(g_5) \]

\[
G_2(s) = \frac{10}{s+10} = \frac{1}{0.1s+1}
\]

\[
G_4(s) = \frac{10(s+1.1)}{1.1(s+1)(s+10)} = \frac{(0.909s+1)}{(s+1)(0.1s+1)}
\]

\[
G_4(s) = \frac{10(s+4)}{4(s+1)(s+10)} = \frac{(0.25s+1)}{(s+1)(0.1s+1)}
\]

\[
G_3(s) = \frac{10}{(s+1)(s+10)} = \frac{1}{(s+1)(0.1s+1)}
\]

Effect of closed-loop poles on response
Effect of closed-loop poles on response

OCTAVE Program

g1=zp([],[-1+i*0.5 -1-i*0.5],1.25);
    step(g1)
    hold on

hold on

step(g2)
    hold on

g3=zp([],[-1+i*2 -1-i*2],5);
    step(g3)
    hold on

step(g4)

\[ \zeta = 0.45 \]
\[ \zeta = 0.707 \]
\[ \zeta = 0.9 \]

Pole at -10

<table>
<thead>
<tr>
<th>-4</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
</tbody>
</table>
A system (linear or non-linear) is said to be BIBO (bounded input, bounded output) stable if, for every bounded input, the output is bounded for all time.

An LTI (Linear Time-Invariant) system must have all poles in the left-half of the s-plane (negative real parts) for it to be stable.

In other words, the roots of the characteristic equation must all have negative real parts.

If a pole, or poles, lie on the imaginary axis, the system is critically, or limitedly, stable.

If a linear system is unstable, even in the absence of any input, the output will grow without bounds and becomes infinitely large as time goes to infinity.
A system is stable if all the roots of the system’s characteristic equation \( 1 + G(s)H(s) = 0 \) have negative real parts.

The problem is if the characteristic equation is of an order higher than two, it is not easy to find the roots. (Of course, there are computer programs, e.g. MATLAB or OCTAVE, that helps with this.)

Fortunately, there is a simple criterion, known as Routh’s Stability Criterion (sometimes also known as the Routh-Hurwitz Stability Criterion), which enables us to find out the number of roots of the characteristic equation that lie on the right-half of the s-plane, i.e. have positive real parts, without having to factor the characteristic polynomial.
Routh’s Stability Criterion

Procedure

1) Form the characteristic equation

\[ a_0 S^n + a_1 S^{n-1} + a_2 S^{n-2} + \cdots + a_{n-1} S + a_n = 0 \quad a_0 > 0 \]

We assume that \( a_n \neq 0 \); i.e. any zero roots have been removed.

Example: \( 6s^5 + 3s^4 + 5s^3 + 10s^2 = 0 \) or \( s^2 (6s^3 + 3s^2 + 5s + 10) = 0 \)

use the equation \( 6s^3 + 3s^2 + 5s + 10 = 0 \)

2) If any of the coefficients is negative or zero, the system is not stable.

3) If all the coefficients are positive, there is still no guarantee that all the roots have negative real parts. We then form the Routh Array and use the Routh Criterion to determine the number of roots with positive real parts.
## Routh’s Stability Criterion

**Characteristic equation**

\[ a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0 \quad a_0 > 0 \]

**Routh Array**

<table>
<thead>
<tr>
<th>( s^n )</th>
<th>( a_0 )</th>
<th>( a_2 )</th>
<th>( a_4 )</th>
<th>( a_6 )</th>
<th>( \cdots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s^{n-1} )</td>
<td>( a_1 )</td>
<td>( a_3 )</td>
<td>( a_5 )</td>
<td>( a_7 )</td>
<td>( \cdots )</td>
</tr>
<tr>
<td>( s^{n-2} )</td>
<td>( )</td>
<td>( )</td>
<td>( )</td>
<td>( )</td>
<td>( )</td>
</tr>
<tr>
<td>( s^{n-3} )</td>
<td>( )</td>
<td>( )</td>
<td>( )</td>
<td>( )</td>
<td>( )</td>
</tr>
<tr>
<td>( \cdots )</td>
<td>( )</td>
<td>( )</td>
<td>( )</td>
<td>( )</td>
<td>( )</td>
</tr>
<tr>
<td>( s^1 )</td>
<td>( )</td>
<td>( )</td>
<td>( )</td>
<td>( )</td>
<td>( )</td>
</tr>
<tr>
<td>( s^0 )</td>
<td>( )</td>
<td>( )</td>
<td>( )</td>
<td>( )</td>
<td>( )</td>
</tr>
</tbody>
</table>
### Routh’s Stability Criterion

**Characteristic equation**

\[ a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0 \quad a_0 > 0 \]

### Routh Array

<table>
<thead>
<tr>
<th>( s^n )</th>
<th>( a_0 )</th>
<th>( a_2 )</th>
<th>( a_4 )</th>
<th>( a_6 )</th>
<th>( \cdots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s^{n-1} )</td>
<td>( a_1 )</td>
<td>( a_3 )</td>
<td>( a_5 )</td>
<td>( a_7 )</td>
<td>( \cdots )</td>
</tr>
<tr>
<td>( s^{n-2} )</td>
<td>( b_1 )</td>
<td>( b_2 )</td>
<td>( b_3 )</td>
<td>( b_4 )</td>
<td>( \cdots )</td>
</tr>
<tr>
<td>( s^{n-3} )</td>
<td>( b_3 )</td>
<td>( b_4 )</td>
<td>( b_5 )</td>
<td>( b_6 )</td>
<td>( \cdots )</td>
</tr>
<tr>
<td>( \cdots )</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
</tr>
<tr>
<td>( s^1 )</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
</tr>
<tr>
<td>( s^0 )</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
</tr>
</tbody>
</table>

1. \( b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \)
2. \( b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} \)
3. \( b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1} \)
In developing the array, an entire row can be multiplied by a positive number to simplify the process without affecting the result.

Routh’s Stability Criterion states that the number of roots with positive real parts is equal to the number of changes in sign of the coefficients in the first column of the array.
**Example**

Determine the conditions for the following equation to have only roots with negative real parts.

\[ a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0 \]

**Routh Array**

\[
\begin{array}{ccc}
 s^3 & a_0 & a_2 \\
 s^2 & a_1 & a_3 \\
 s^1 & a_1 a_2 - a_0 a_3 \\
 s^0 & a_1 & a_3 \\
\end{array}
\]

If \[ a_1 a_2 - a_0 a_3 > 0 \], then there is no sign change and there is no roots with positive real parts.

If \[ a_1 a_2 - a_0 a_3 < 0 \], then there are two sign changes. Therefore there are two roots with positive real parts.
Routh’s Stability Criterion

Special case 1

A zero occurs in the first column of any row while the remaining terms are not zero, or there is no remaining term. Solution: The zero term is replaced by a small positive number and the array is processed accordingly.

Example

\[ s^3 + 3s^2 + s + 3 = 0 \]

Routh Array

| \( s^3 \) | \( 1 \) | \( 1 \) |
| \( s^2 \) | \( 3 \) | \( 3 \) |
| \( s^1 \) | \( 0 \rightarrow \epsilon \) |
| \( s^0 \) | \( 3 \) |

- If the sign of the coefficient in the row above is the same as that below (as in this case), then there are a pair of imaginary roots.
- If the sign of the coefficient in the row above is different from that below, there is one sign change indicating one root with positive real part.
Special case 2

If all the coefficients, or the only one coefficient, in a derived row are zero, it means that there are roots of equal magnitude located symmetrically about the origin. Example: The characteristic equation have factors such as $(s + \sigma)(s - \sigma)$ or $(s + j\omega)(s - j\omega)$.

For such cases, form an auxiliary polynomial with the coefficients of the row above the all-zero row and using the coefficients of the derivative of this polynomial to replace the all-zero row.
Routh’s Stability Criterion

Special case 2 – Example

\[ s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0 \]

Note that because not all the coefficients are positive, this indicates that there is at least one root with a positive real part.

Routh Array

| \( s^5 \) | 1 | 24 | -25 |
| \( s^4 \) | 2 | 48 | -50 |
| \( s^3 \) | 0 | 0 |   |

Use this as auxiliary polynomial

\[ P(s) = 2s^4 + 48s^2 - 50 \]

\[ \dot{P}(s) = 8s^3 + 96s \]

New Routh Array

| \( s^5 \) | 1 | 24 | -25 |
| \( s^4 \) | 2 | 48 | -50 |
| \( s^3 \) | 8 | 96 |   |
| \( s^2 \) | 24 | -50 |   |
| \( s^1 \) | 112.7 | 0 |   |
| \( s^0 \) | -50 |   |   |

There is one change in sign in the first column – one root with +ve real part.
End