An Enhanced Ray-Shooting Approach to Force-Closure Problems

Force-closure is a fundamental topic in grasping research. Relevant problems include force-closure test, quality evaluation, and grasp planning. Implementing the well-known force-closure condition that the origin of the wrench space lies in the interior of the convex hull of primitive wrenches, Liu presented a ray-shooting approach to force-closure test. Because of its high efficiency in 3D work space and no limitation on the contact number of a grasp, this approach is advanced. Achieving some new results of convex analysis, this paper enhances the above approach in three aspects. (a) The exactness is completed. In order to avoid trouble or mistakes, the dimension of the convex hull of primitive wrenches is taken into account, which is always ignored until now. (b) The efficiency is increased. A shortcut which skips some steps of the original force-closure test is found. (c) The scope is extended. Our simplified ray-shooting approach yields a grasp stability index suitable for grasp planning. Numerical examples in fixtureing and grasping show the enhancement superiority. [DOI: 10.1115/1.2336259]

Keywords: force-closure, grasp planning, multifingered robot hand, ray-shooting approach

1 Introduction

Multifingered robotic grasping has been ardently studied since the pioneer work of Salisbury and Roth [1]. Force-closure is a fundamental topic in grasping research. This property means the capability of a grasp to equilibrate any external wrench and to restrain any motion on the grasped object. It is a prerequisite to stable grasping. Force-closure problems mainly include:

- Force-closure test: given contact positions on an object, determine if the grasp is force-closure.
- Grasp quality evaluation: given contact positions on an object, evaluate the closure quality of the grasp by a performance index.
- Optimal grasp planning: given an object, determine the contact positions to construct a force-closure grasp with optimal performance quality.

These problems can be discussed in the wrench space [1–5], the contact force space, [6–10], or their dual spaces [10–13]. The wrench space and its dual space are 6D vector spaces, while the dimensions of the contact force space and its dual space are both \( m_0 + 3m_f + 4m_s \), where \( m_0, m_f, \) and \( m_s \) are numbers of frictionless point contacts, frictional point contacts, and soft finger contacts, respectively.

1.1 Related Work. Investigation in the wrench space tells that a grasp is force-closure if and only if the primitive wrenches positively span the entire wrench space [1], or equivalently, the origin of the wrench space is an interior point of the convex hull of the primitive wrenches [2]. By implementing this condition, after the 2D test [3] Liu [4] presented a ray-shooting based algorithm for 3D. Zhu et al. [5] proposed a generally applicable algorithm without linearizing the friction cones. In the contact force space, Murray et al. [6] revealed that a grasp is force-closure if and only if the grasp matrix is surjective and there is a strictly internal force. Various forms of this condition can be found in Refs. [7–10]. Zuo and Qian [7] extended the condition to soft multifingered grasps. Following Buss et al. [8], Han et al. [9] formulated force-closure test as a convex optimization problem involving linear matrix inequalities. Bicchi [10] took account of the kinematics of the grasping mechanism. Recently, Zheng and Qian [13] generalized the method of form-closure analysis [10–12] to force-closure by the duality between the infinitesimal motion and the wrench.

As a higher topic than qualitative test, quantitative evaluation indicates the goodness of various grasps. Optimal grasp planning cannot proceed without it. Li and Sastry [14] presented three quality measures: the smallest singular value and the volume of the grasp matrix as well as a task-oriented measure. Zuo and Qian [7], Buss et al. [8], evaluated the stability of a grasp by its extent to satisfy the friction constraints. Zheng and Qian [13] explored the tolerance of force-closure grasps to some grasping uncertainties. Kirkpatrick et al. [15], Ferrari and Camny [16] assessed the “efficiency” by the radius of the largest ball centered at the origin of the wrench space, contained in the convex hull of the primitive wrenches. Using the \( Q \) distance, Zhu and Wang [17] made it possible to compute the measure of efficiency for the first time. Other quality measures were proposed by Varma and Tasch [18], Xiong and Xiong [19], Salunkhe et al. [20].

An early stage of optimal grasp planning research focused on synthesizing force-closure grasps on simple objects with limited contacts. On polygonal objects, Nguyen [21] computed independent regions for two frictional or four frictionless point contacts to achieve a force-closure grasp. Markenscoff and Papadimitriou [22] proposed an analytic method for calculating the optimum grip. Park and Starr [23] built a 3-finger grasp, while Tung and Kak [24] fast constructed a 2-finger one. On irregular 2D and 3D objects, Chen and Burdick [25] considered 2-finger antipodal point grasps. Li et al. [26] developed a geometrical algorithm for computing 3-finger force-closure grasps. Ponce and co-workers [27–29] extended Nguyen’s [21] idea to 2-finger, 3-finger, and 4-finger force-closure grasps on 2D curved, polygonal, and polyhedral objects, respectively. In the recent years, limitation on the contact number has been eliminated. Liu [30] calculated \( n \)-finger grasps on polygons. Ding et al. [31] considered 3D \( n \)-finger grasps.
whose $k$ fingers have been located in advance. Based on the $Q$ distance, Zhu and Wang [17] planned optimal grasps on 3D objects with curved surfaces. With the ray-shooting based algorithm [4], Liu et al. [32] sought force-closure grasps on objects in the discrete domain. In addition, the algorithm for fixture design can be applied to grasp planning as well [33].

### 1.2 Our Work
In the previous work, we are especially interested in the ray-shooting approach [4]. Since Mishra et al. [2] proposed the force-closure condition, no algorithm implemented it on 3D grasps during the succeeding twelve years until Liu put forward the ray-shooting based algorithm [4]. Up to the present, it is still the fastest way to force-closure test and frequently used in grasp planning as well as fixture design [31–33]. After repeatedly studying his work, we first found a shortcut to simplify its application to force-closure test [34]. Second, we discover that, although the test algorithm is valid in most cases, it may commit errors because the dimension of the convex hull of primitive wrenches is ignored. When the convex hull is below 6D, the linear programming (LP) formulation for solving the ray-shooting problem will be unbounded, or the origin will be mistaken for an interior point of the convex hull. This motivates us to investigate the dimension and the relative interior of a convex set. According to the relative interior as well as dimension rather than its interior. As the mathematics of the convex hull and its dimension are related strongly, we classify the convex hulls into four categories. A grasp is force-closure to the dimensions and the relative interior point, so that the origin lies in its interior. The former condition is equivalent to that the grasp matrix has full row rank and a certain linear system is consistent. The latter can be determined by the simplified ray-shooting approach. The consistency of the linear system ensures that our LP formulation is bounded and always has solution. Third, the simplified ray-shooting approach turns out a grasp stability index, which is relevant to the inclination angles of contact forces. It has different meaning from the quality indices [15–17], which reflect the resultant wrenches generated by a grasp. Compared with other formulations of grasp stability [7,8], ours is easier to compute. Furthermore, the original ray-shooting approach [4,32] does not yield such an index, so the index is applied to optimal grasp planning of arbitrary 3D objects for the first time. Needless to say, all the above start from Liu’s trailblazing work [4]. In addition, for use in derivation, we deduce a number of theorems of convex analysis. Some of them are brand new.

### 2 Preliminaries
Our work is based on the following assumptions:

1. The fingers and the grasped object are rigid bodies. Like Refs. [1–34], we do not consider their compliance and contact region deformation as Refs. [35–37], etc. All contacts are point-to-point hard contacts.
2. Each finger contacts the object at a regular point, where $\mathbf{n}_i$, $\mathbf{o}_i$, and $\mathbf{t}_i$ are well defined.
3. The finger number $m \geq 3$, which is a prerequisite for achieving 3D force-closure.

Consider an $m$-finger robot hand grasping a 3D object, fixed with a right-handed coordinate frame. The contact force $\mathbf{f}_i$ at contact $i$ can be expressed in the local coordinate frame $\{\mathbf{n}_i, \mathbf{o}_i, \mathbf{t}_i\}$ by

$$
\mathbf{f}_i = [f_{i_0} f_{i_1} f_{i_2}]^T
$$

(1)

To avoid separation and slippage at contact, $\mathbf{f}_i$ must satisfy

$$
f_{i_0} \geq 0, \quad f_{i_0}^2 + f_{i_1}^2 + f_{i_2}^2 \leq \mu^2 f_{i_0}^2
$$

The above nonlinear contact constraint defines a circular cone called a friction cone. For simplicity, we substitute an $n$-sided polyhedral cone for it ($n \geq 3$ since the friction cone is 3D), as shown in Fig. 1. The side edges are expressed in the frame $\{\mathbf{n}_i, \mathbf{o}_i, \mathbf{t}_i\}$ by

$$
\mathbf{s}_j = [1 \mu \cos(2j\pi/n) \mu \sin(2j\pi/n)]^T
$$

Thus $\mathbf{f}_i$ in the friction cone can be approximately represented by

$$
\mathbf{f}_i = \sum_{j=1}^{n} \alpha_{ij} \mathbf{s}_j, \quad \alpha_{ij} \geq 0
$$

(3)

From Eqs. (1)–(3), $f_{i_0}$, $f_{i_1}$, and $f_{i_2}$ are specified by

$$
f_{i_0} = \sum_{j=1}^{n} \alpha_{ij} f_{i_2}, \quad f_{i_1} = \mu \sum_{j=1}^{n} \alpha_{ij} \cos(2j\pi/n), \quad f_{i_2} = \mu \sum_{j=1}^{n} \alpha_{ij} \sin(2j\pi/n)
$$

(4)

The wrench (a couple of force and moment applied at the origin of the object coordinate frame) produced by $\mathbf{f}_i$ is given by

$$
\mathbf{w}_i = \mathbf{G}_i \mathbf{f}_i
$$

(5)

where

$$
\mathbf{G}_i = \begin{bmatrix}
\mathbf{n}_i & \mathbf{0}_i & \mathbf{t}_i \\
\mathbf{r}_i \times \mathbf{n}_i & \mathbf{r}_i \times \mathbf{0}_i & \mathbf{r}_i \times \mathbf{t}_i
\end{bmatrix}
$$

Substituting Eq. (3) into Eq. (5) yields

$$
\mathbf{w}_i = \sum_{j=1}^{n} \alpha_{ij} \mathbf{w}_{ij}, \quad \alpha_{ij} \geq 0
$$

(6)

where

$$
\mathbf{w}_{ij} = \mathbf{G}_i \mathbf{s}_j, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n
$$

The vector $\mathbf{w}_{ij}$ is called a primitive wrench. Thereby the resultant wrench applied by the hand is

$$
\mathbf{w} = \sum_{i=1}^{m} \mathbf{w}_i = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} \mathbf{w}_{ij}, \quad \alpha_{ij} \geq 0
$$

A grasp is said to be force-closure if there always exist nonnegative reals $\alpha_{ij}, i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ such that $-\mathbf{w}_{\text{ext}} = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} \mathbf{w}_{ij}$ for any $\mathbf{w}_{\text{ext}} \in \mathbb{R}^n$, which is equivalent to that the primitive wrenches positively span the whole wrench space [1]. Let $\mathcal{W}$ be the convex hull of the primitive wrenches:

$$
\mathcal{W} = \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} \mathbf{w}_{ij} \mid \sum_{j=1}^{n} \alpha_{ij} = 1, \quad \alpha_{ij} \geq 0 \right\}
$$

(7)

Not only the force-closure property but also the grasp quality can be revealed from $\mathcal{W}$ [2,4,16,17]. Noticing that $\mathcal{W}$ may be 6D or of lower dimension, in general we would discuss its relative interior as well as dimension rather than its interior. As the math-
ematical basis, we first extend the theorems on interior in convex analysis [38] to relative interior. Later, all of them will be used to solve the foregoing force-closure problems.

3 Results of Convex Analysis

Let $S$ be a nonempty convex set in $\mathbb{R}^d$. The dimension of $S$ is the dimension of its affine hull $\text{aff}S$, namely the dimension of the corresponding parallel subspace. The relative interior of $S$ is the interior of $S$ relative to aff $S$. In fact, the definition "relative interior" is an extension of the definition "interior." When aff $S=\mathbb{R}^d$, ri $S=\text{int}S$. As the former covers the latter, and we can regard the latter as a special case of the former. The set $\text{cl}S\cap \text{ri}S$ is called the relative boundary of $S$, denoted by $rbS$.

**Theorem 1.** $\text{int}S$ if and only if dim $S=d$ and $0 \in \text{ri}S$.

**Theorem 2.** If $S$ is the convex hull of a finite set of points $x_1, x_2, \ldots, x_n$ in $\mathbb{R}^d$, i.e., $S=\text{conv}\{x_1, x_2, \ldots, x_n\}$, then any strictly positive convex combination of $x_1, x_2, \ldots, x_n$ is a relative interior point of $S$, i.e., $\sum_{k=1}^{n} \lambda_k x_k \in \text{ri}S$ for any $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ with $\sum_{k=1}^{n} \lambda_k=1$.

To prove this theorem, we need two lemmas.

**Lemma 1.** If $S$ is the convex hull of affinely independent points $x_1, x_2, \ldots, x_n$ in $\mathbb{R}^d$, i.e., $S=\text{conv}\{x_1, x_2, \ldots, x_n\}$, then any strictly positive convex combination of $x_1, x_2, \ldots, x_n$ is a relative interior point of $S$, i.e., $\sum_{k=1}^{n} \lambda_k x_k \in \text{ri}S$ for any $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ with $\sum_{k=1}^{n} \lambda_k=1$.

**Proof.** Since $x_1, x_2, \ldots, x_n$ are affinely independent, they constitute an affine basis of aff $S$; hence each point $x \in \text{aff}S$ can be expressed by $x = \sum_{k=1}^{n} \lambda_k x_k$ with $\sum_{k=1}^{n} \lambda_k = 1$, and the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ are unique. Therefore, we define $\varphi: \text{aff}S \rightarrow \mathbb{R}^n$ by letting

$$
\varphi \left( \sum_{k=1}^{n} \lambda_k x_k \right) = [\lambda_1, \lambda_2, \ldots, \lambda_n]^T \text{ with } \sum_{k=1}^{n} \lambda_k = 1,
$$

This is an affine mapping; in particular, it is continuous. Let

$$
H_k = (\{[\lambda_1, \lambda_2, \ldots, \lambda_n] : \sum_{k=1}^{n} \lambda_k > 0\}, \text{ for } k = 1, 2, \ldots, e).
$$

Then $H_1, H_2, \ldots, H_e$ are open halfspaces in $\mathbb{R}^e$; hence, by continuity, $\varphi^{-1}(H_1), \varphi^{-1}(H_2), \ldots, \varphi^{-1}(H_e)$ are open in $\text{aff}S$. Their intersection

$$
\bigcap_{k=1}^{e} \varphi^{-1}(H_k) = \left\{ \sum_{k=1}^{e} \lambda_k x_k : \lambda_1, \lambda_2, \ldots, \lambda_e > 0, \sum_{k=1}^{e} \lambda_k = 1 \right\}
$$

is therefore also open in $\text{aff}S$. This in particular shows that the set $\bigcap_{k=1}^{e} \varphi^{-1}(H_k)$ is nonempty. Since $\bigcap_{k=1}^{e} \varphi^{-1}(H_k)$ is a set of positive convex combinations of $x_1, x_2, \ldots, x_n$, we have $\bigcap_{k=1}^{e} \varphi^{-1}(H_k) \subseteq S$. In other words, $S$ contains a nonempty set which is open in $\text{aff}S$, whence $\bigcap_{k=1}^{e} \varphi^{-1}(H_k) \subseteq S$.

**Lemma 2.** If $x_0 \in \text{ri}S$ and $x_1 \in S$, then $(1-\lambda)x_0 + \lambda x_1 \in \text{ri}S$ for all $\lambda \in [0, 1]$ [38].

**Proof of Theorem 2.** Let $h = \dim S-\dim(\text{aff}S)$. Then there exists $h+1$ affinely independent points from $\{x_1, x_2, \ldots, x_n\}$ without loss of generality, say $x_1, x_2, \ldots, x_{h+1}$. Let

$$
S_1 = \text{conv}\{x_1, x_2, \ldots, x_{h+1}\} \text{ and } S_2 = \text{conv}\{x_{h+2}, x_{h+3}, \ldots, x_n\}.
$$

Then $S_1, S_2 \subseteq S$ and dim(aff $S_1$) = dim $S_1$ = h. Let

$$
x = \sum_{k=1}^{h} \lambda_k x_k \text{ with } \lambda_1, \lambda_2, \ldots, \lambda_h > 0 \text{ and } \sum_{k=1}^{h} \lambda_k = 1.
$$

This equation can be rewritten as

$$
x = l_1 \sum_{k=1}^{h-1} \lambda_k x_k + l_2 \sum_{k=h+2}^{e} \lambda_k x_k = l_1 y + l_2 z,
$$

where $l_1 = \sum_{k=1}^{h-1} \lambda_k$, $l_2 = \sum_{k=h+2}^{e} \lambda_k$, $y = \sum_{k=1}^{h-1} \lambda_k x_k / l_1$, and $z = \sum_{k=h+2}^{e} \lambda_k x_k / l_2$. Note that $y$ is a strictly positive convex combination of $x_1, x_2, \ldots, x_{h+1}$, and from Lemma 1, $y \in \text{ri}S_1$. From aff $S_1 \subseteq \text{aff}S$ and dim(aff $S_1$) = dim(aff $S$), we have aff $S_1 \subseteq \text{aff}S$. But since $S_1 \subseteq S$, it follows that $y \in S$. In addition, $z$ is a convex combination of $x_{h+2}, x_{h+3}, \ldots, x_n$; thus $z \in S_2$, which implies $z \in S$. Hence $S_1 \subseteq \text{aff}S$ and $S_2 \subseteq (S' \cap \text{aff}S) + (\text{aff}S)^\perp$. Therefore, $S' = (S' \cap \text{aff}S) + (\text{aff}S)^\perp$, as shown in Fig. 2.

In addition, if $S$ is a compact convex set in $\mathbb{R}^d$ with $0 \in \text{int}S$, $S'$ is a compact convex set in $\mathbb{R}^d$ with $0 \in \text{int}S'$ [38]. In general, if $S$ is compact and $0 \in \text{ri}S$, $S' \cap \text{aff}S$ is a compact convex set in aff $S$ and $0$ is an interior point of $S'$ relative to aff $S$. From $S = (S' \cap \text{aff}S) + (\text{aff}S)^\perp$, thus $S'$ = aff $S$. Therefore, $S' = (S' \cap \text{aff}S) + (\text{aff}S)^\perp$.

**The Support Function** $p_S$ [38] of $S$ is the real-valued function defined by

$$
p_S(x) = \sup_{z \in S} \langle z^T x \rangle
$$

for all $x \in \mathbb{R}^d$ for which the supremum is finite. Let $p_S$ be the support function of $S'$. In the following, we assume that $S$ is a nonempty compact convex set with $0 \in \text{ri}S$.

**Theorem 4.** $p_S(0)$ is finite if and only if $0 \in \text{aff}S$. Moreover, if $0 \in \text{aff}S$, then $p_S(0) = p_S(0 \cap \text{int}S)$ and there is $y \in S' \cap \text{aff}S$ such that $p_S(0) = z^T y$. If $z$ is nonzero, then $p_S(0) = 0$.

**Proof.** From Theorem 3 it follows that any $y \in S'$ can be decomposed into $y_1 \in S' \cap \text{aff}S$ and $y_2 \in (\text{aff}S)^\perp$, thus $z^T y = z^T y_1 + z^T y_2$ for any $y \in S'$. If $0 \in \text{aff}S$, then $z^T y = z^T y_1$, which implies that $p_S(0) = p_S(0 \cap \text{int}S)$. Due to the compactness of $S' \cap \text{aff}S$ (see Theorem 3) and the continuity of the inner product, $p_S(0)$ is finite and there exists a point $y_1 \in S' \cap \text{aff}S$ such that $p_S(0) = z^T y_1$. Conversely, if $0 \in \text{aff}S$, then $z$ can be decomposed into $z_1 \in \text{aff}S$ and $z_2 \in (\text{aff}S)^\perp$, where $z_2$ is nonzero, so $z^T y = z_1^T y_1 + z_2^T y_2$. Since $z_1^T y_1$...
is finite for all $y_1 \in S^c \cap \text{aff } S$ while the supremum of $z^Ty_2$ on $y_2 \in (\text{aff } S)^c$ takes on arbitrarily large values, the supremum of $z^Ty$ on $y \in S^c$ is infinite. This proof can be illustrated in Fig. 2. Furthermore, from Theorem 3, $0 \in \text{int } S$. Then for any nonzero $z \in \text{aff } S$, we may always find $y \in S^c$ such that $z^Ty > 0$; thus $p_S(z) > 0$.

THEOREM 5. Let $z$ be a point in $\text{aff } S$ other than $0$. Then $p_S(z)^{-1}z \in \text{rb } S$.

Proof. From the definition of $p_S$, $z^Ty \leq p_S(z)$ for all $y \in S^c$. From Theorem 4, $p_S^c(z) > 0$. Then $y^2p_S(z)^{-1}z \leq 1$ for all $y \in S^c$, which means $p_S^c(z)^{-1}z \in (S^c)^c$. Since $S$ is closed and contains 0, we have $(S^c)^c = S$ [38], and $p_S^c(z)^{-1}z \in S$. Let $B(p_S^c(z)^{-1}z, r)$ be a closed ball in aff $S$ of radius $r > 0$ centered at $p_S^c(z)^{-1}z$. Let $x = p_S^c(z)^{-1}z + rz/|z|$. Obviously, $x \in B(p_S^c(z)^{-1}z, r)$. Because $y^2x = 1 + rp_S^c(z)/|z| > 1$ where $y \in S^c \cap \text{aff } S$ such that $p_S(z) = z^Ty$ (see Theorem 4), we have $x \in (S^c)^c$, i.e., $x \in S$. Hence $p_S^c(z)^{-1}z \in \text{ri } S$, and $p_S^c(z)^{-1}z \in \text{rb } S$. This proof is illustrated in Fig. 3.

THEOREM 6. Let $z$ be a point in aff $S$. Then $z \in \text{ri } S$ if and only if $p_S(z) < 1$.

Proof. Suppose that $z$ is a point other than 0. Theorem 5 asserts that $p_S^c(z)^{-1}z \in \text{rb } S$. If $p_S(z) < 1$, $z$ is strictly between 0 and $p_S^c(z)^{-1}z$, and from Lemma 2 we obtain $z \in \text{ri } S$. Theorem 4 affirms that $p_S(z) > 0$. If $p_S(z) \geq 1$, then $z$ lies on the relative boundary of $S$ or outside $S$. When $z$ is just 0, $p_S(z) = 0 < 1$. Conversely, $p_S(z) = 0$ implies $z = 0 \in \text{ri } S$.

4 Force-Closure Conditions and Test

4.1 Classification of $\mathcal{W}$. The convex hull of the primitive wrenches, denoted by $\mathcal{W}$, has been used in force-closure analysis for a long time. However, its dimension was always neglected or assumed to be 6 in the 3D work space. In fact, the dimension of $\mathcal{W}$ may be less than 6.

By the dimension and the relative position to the origin 0, we classify $\mathcal{W}$ into four categories:

(a) $\dim \mathcal{W} < 6$ and $0 \in \text{aff } \mathcal{W}$ (Fig. 4(a)).
(b) $\dim \mathcal{W} < 6$ and $0 \notin \text{aff } \mathcal{W}$ (Fig. 4(b)).
(c) $\dim \mathcal{W} = 6$ and $0 \in \text{ri } \mathcal{W}$ (Fig. 4(c)).
(d) $\dim \mathcal{W} = 6$ and $0 \notin \text{ri } \mathcal{W}$ (Fig. 4(d)).

When ignoring the dimension of $\mathcal{W}$, the ray-shooting based algorithm [4] does not have a solution in case (a), and mistakes 0 for an interior point of $\mathcal{W}$ in case (b) if $0 \in \text{ri } \mathcal{W}$. Hence, in what follows, we take into account the dimension of $\mathcal{W}$ and suggest a way to avoid these errors.

4.2 Force-Closure Conditions. Our conditions originate from Mishra et al. [2], who wrote:

PROPOSITION 1. A grasp is force-closure if and only if $0 \in \text{int } \mathcal{W}$.

Theorem 1 gives directly:

PROPOSITION 2. $0 \in \text{int } \mathcal{W}$ if and only if $\dim \mathcal{W} = 6$ and $0 \in \text{ri } \mathcal{W}$ (Fig. 4(d)).

The convex cone determined by $\mathcal{W}$ and 0 consists of the resultant wrenches that can be generated by the grasp. If $\dim \mathcal{W} < 6$ and $0 \in \text{ri } \mathcal{W}$, then the grasp can generate resultant wrenches in a proper subspace of $\mathbb{R}^6$, namely aff $\mathcal{W}$, as depicted in Fig. 4(b). Such a grasp is said to be partially force-closure [10]. Let $w_0$ be the centroid of the primitive wrenches $w_i$ and $T$ the translate of $\mathcal{W}$ by $-w_c$.

\begin{equation}
\begin{aligned}
w_c &= \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n w_{ij} \\
T &= \mathcal{W} - w_c
\end{aligned}
\end{equation}

From Eq. (7) and Theorem 2 it follows that $w_0 \in \text{ri } \mathcal{W}$. Thus $T$ is a compact convex set in $\mathbb{R}^6$ with $0 \in \text{ri } T$, and aff $T$ is a subspace of $\mathbb{R}^6$. Furthermore, we readily have

PROPOSITION 3. The following statements are true: (1) $\dim \mathcal{W} = 6$ if and only if $\dim T = 6$; (2) $0 \in \text{ri } \mathcal{W}$ if and only if $-w_c \in \text{ri } T$.

According to Proposition 3, the properties of $\mathcal{W}$ can be investigated from $T$. Let

\begin{equation}
\begin{aligned}
T &= [w_{11} - w_c, \ldots, w_{ij} - w_c, \ldots, w_{nn} - w_c] \in \mathbb{R}^{6 \times mn}
\end{aligned}
\end{equation}

PROPOSITION 4. $\dim T = 6$ if and only if $\text{rank } T = 6$.

Proof. As aff $T$ is a subspace of $\mathbb{R}^6$ and is equal to the range of the matrix $T$, the dimension of aff $T$ equals the rank of $T$, i.e., $\dim \text{aff } T = \text{rank } T$.

Let $\mathcal{T}$ be the polar set of $T$. From Eqs. (7), (8), and (11), $\mathcal{T}$ can be expressed by

\begin{equation}
\begin{aligned}
\mathcal{T} &= \{u \in \mathbb{R}^6 | (w_{ij} - w_c)^Tu \leq 1, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}
\end{aligned}
\end{equation}

Let $p$ denote the support function of $\mathcal{T}$:

\begin{equation}
p(w) = \sup_{u \in \mathcal{T}} w^Tu
\end{equation}

for all $w \in \mathbb{R}^6$ for which the supremum is finite.

PROPOSITION 5. $-w_c \in T$ if and only if $p(-w_c) < 1$.

Proof. If $p(-w_c) < 1$, then from Theorem 4 we have $-w_c$
and only if $p(-w_c) < 1$.

The above force-closure conditions and their relations are summarized in Fig. 5. In addition, it especially needs to be cautious about:

1. $\dim W=6$ implies $\text{rank } G=6$, but the converse does not hold true. The dimension of $W$ is equal to the rank of $T$ and may be less than the rank of $G$.
2. $p(-w_c) < 1$ implies $-w_c \in \text{aff } T$, but $-w_c \in \text{aff } T$ must be confirmed prior to computing $p(-w_c)$. $p(-w_c)$ is finite and can be computed only if $-w_c \in \text{aff } T$.

4.3 Force-Closure Test Algorithm. Referring to Fig. 5, the force-closure test can be formulated as:

Step 1: Calculate the primitive wrenches $w_{ij}$ by Eqs. (2) and (6).

Step 2: Compute the centroid $w_c$ by Eq. (10).

Step 3: Construct the matrix $T$ by Eq. (12) and calculate its rank.

Step 4: Determine if $-w_c \in \text{aff } T$. Since aff $T$ equals the range of $T$, $-w_c \in \text{aff } T$ if and only if the linear system $Tz=-w_c$ is consistent, or $\|TT^Tw_c-w_c\|=0$, where $T^*$ is the psuedoinverse of $T$. If $\|TT^Tw_c-w_c\|=0$, then go to Step 5; otherwise the algorithm terminates.

Step 5: Compute $p(-w_c)$. From Eqs. (13) and (14), it is formulated as an LP problem:

Maximize $-w^Tc$

subject to $(w_{ij}-w_c)^Tc_i \leq 1, i=1,2,\ldots,m, j=1,2,\ldots,n$

(15)

The algorithm ends.

The algorithm turns out four types of results corresponding to the foregoing categories of $\mathcal{Y}^W$:

(a) $\text{rank } T<6$ and $\|TT^Tw_c-w_c\| \neq 0$.
(b) $\text{rank } T<6$ and $\|TT^Tw_c-w_c\|=0$.
(c) $\text{rank } T=6$ and $p(-w_c)=1$.
(d) $\text{rank } T=6$ and $p(-w_c) < 1$ (force-closure).

This formulation of force-closure test is closely related to the typical ray-shooting problem [4], i.e., a problem of finding the intersection of a ray with the boundary of a polytope. Denote the ray from the point $w_c$ to the origin $0$ by $R = \{-\beta w_c + w_c | \beta \geq 0\}$, where $w_c \neq 0$. The condition $-w_c \in \text{aff } T$, determined by $\|TT^Tw_c-w_c\|=0$, ensures that $R$ is contained in aff $T$ and intersects with $\text{rb } \mathcal{Y}$, which in turn guarantees that $p(-w_c)$ is finite. Hence, since $T^*$ is nonempty, the LP problem (15) always has solution and can be solved in $O(mn)$ time. In fact, intersection happens at the point $p(-w_c)^{-1}w_c + w_c$, as shown in Fig. 6. Then $p(-w_c)$ equals the ratio of the distance between $w_c$ and $0$ to the one between $w_c$ and the intersection point. Therefore, $p(-w_c) < 1$ means $0 \in \text{ri } \mathcal{Y}$. This geometric insight into $p(-w_c)$ first helps us skip the computation of the distances and simplify the typical ray-shooting approach to force-closure test [4,34]. Moreover, $p(-w_c)$ intuitively suggests a risk of losing force-closure, or its reciprocal gives a safety factor of force-closure.

5 Optimal Grasp Planning

5.1 Performance Index of a Grasp Configuration. The previous section indicates that $p(-w_c)$ can be applied to reflecting the force-closure property of grasps. Hereafter, we give a further physical interpretation of $p(-w_c)$. Assume that $\dim W=6$, and then the force-closure property is entirely represented by $p(-w_c)$.

Equation (10) shows that $w_c$ is the convex combination of $w_{ij}$ with the coefficients

$$
a_{c,ij} = \frac{1}{mn}, \quad i=1,2,\ldots,m, \quad j=1,2,\ldots,n
$$

Substituting Eq. (16) into Eq. (4) with $\sum_{j=1}^m \cos(2j\pi/n) = 0$

$$
f_{c,ij} = f_{c,io} = f_{c,ji} = 0
$$

Hence, $w_c$ is the resultant wrench of the contact forces $f_{c,ij}$,

$$
w_c = \sum_{j=1}^m G_{c,ij}f_{c,ij}
$$

When $p(-w_c) \geq 1$, $0 \in \text{ri } \mathcal{Y}$ and the grasp is not force-closure.
\( p(-w_c) \) implies how far the grasp is away from force-closure.

Let us pay more attention to the case of \( p(-w_c) < 1 \). Then \( \mathbf{0} \in \mathcal{W}_i \) and the affine hull \( \mathcal{W}_i \) is a subspace of \( \mathbb{R}^6 \), which comprises the resultant wrenches that the grasp can apply on the gripped object. From \( w_c \in \mathcal{W}_i \) it follows that \(-w_c \in \mathcal{W}_i \), i.e., there exist non-negative reals \( \lambda_{ij}, \ i=1,2,\ldots,m \) and \( j=1,2,\ldots,n \) such that

\[
-w_c = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij} w_{ij}
\]

This equation can be rewritten as

\[
-w_c = \sigma \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij} w_{ij} = \sigma w_c
\]

where

\[
\sigma = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij}
\]

\[
w_a = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij} w_{ij}
\]

From Eq. (4), \( \sigma \) specifies the sum of the normal force components for all contacts. Equation (20) indicates that \( w_a \in \mathcal{W}_i \) and

\[
\sigma = \frac{\|w_a\|}{\|w_c\|}
\]

From Eqs. (7), (21), and (22) we see that \( w_a \in \mathcal{W}_i \). Thus from Eq. (23), \( \sigma \) attains its minimum value \( \bar{\sigma} \) when \( w_a \) is the intersection point of \( \mathcal{W}_i \) with \( \mathcal{R} \) (Fig. 6(b)), i.e.,

\[
w_a = -p(-w_c)^{-1} w_c + w_c
\]

\[
\sigma = \left\| -p(-w_c)^{-1} w_c + w_c \right\| = \frac{1}{1-p(-w_c)} - 1
\]

The derivative of \( \sigma \) with respect to \( p(-w_c) \) is

\[
\frac{d\sigma}{dp(-w_c)} = \frac{\left[ 1 - p(-w_c) \right]^2}{\left( 1 - p(-w_c) \right)^2} > 0
\]

This means that \( \sigma \) is increasing on \( (0,1) \).

Suppose that \( \alpha_{ij}, i=1,2,\ldots,m \) and \( j=1,2,\ldots,n \) are non-negative coefficients satisfying Eq. (19) with \( \sigma = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} \). Substituting \( \alpha_{ij}, i=1,2,\ldots,n \) into Eq. (3) yields the contact forces \( f_{c,i} \) satisfying

\[
-w_c = \sum_{i=1}^{m} G_i f_{c,i},
\]

\[
f_{c,i} \geq 0,
\]

\[
(f_{c,i,0})^2 + (f_{c,i,0})^2 \leq (\mu f_{c,i,0})^2
\]

\[
\sum_{i=1}^{m} f_{c,i} = \sigma
\]

Let \( f_{\text{int}} = [f_{\text{int},1}^T f_{\text{int},2}^T \cdots f_{\text{int},m}^T]^T \) where

\[
f_{\text{int}} = f_{c,i} + f_{c,i}
\]

Substituting Eq. (17) into Eq. (29) leads to

\[
f_{\text{int},i} = f_{c,i} + \frac{1}{m} f_{\text{int},i}, \quad f_{\text{int},i} = f_{c,i}
\]

Combining Eqs. (18), (25), and (29) indicates

\[
\sum_{i=1}^{m} G_i f_{\text{int},i} = \mathbf{0}
\]

From Eqs. (26), (27), and (30), we obtain

\[
f_{\text{int},i} \geq f_{c,i} \geq 0
\]

\[
f_{\text{int},i}^2 + f_{c,i}^2 \leq (\mu f_{c,i})^2 \leq \mu^2 f_{\text{int},i}^2
\]

Equation (31) means that \( f_{\text{int}} \) is an internal force, while Eqs. (32) and (33) indicate that \( f_{\text{int}} \) is strictly inside the friction cone. Hence \( f_{\text{int}} \) is a strictly internal force [6–10].

Let

\[
\xi_i = \frac{f_{\text{int},i}^2 + f_{c,i}^2}{\mu f_{\text{int},i}^2} \quad \text{for } i = 1, 2, \ldots, m
\]

The value \( \xi_i \) implies the inclination angle of \( f_{\text{int},i} \). Smaller values \( \xi_i, i=1,2,\ldots,m \) mean higher grasp stability [8]. In grasp planning, we hope \( \xi_i \) as small as possible.

Substituting Eqs. (30) and (33) into Eq. (34) yields

\[
\xi_i \leq \bar{\xi}_i
\]

where

\[
\bar{\xi}_i = \frac{f_{c,i}^2}{f_{c,i}^2 + 1/m}
\]

From Eqs. (35) and (36), \( \bar{\xi}_i \) is an upper bound of \( \xi_i \) and increasing on \( f_{c,i} \). Minimizing \( f_{\text{int},i} \) will minimize \( \xi_i \), which in turn reduces \( \xi_i \). However, note that minimization of \( f_{\text{int},i}, i=1,2,\ldots,m \) is a multiobjective optimization problem (MOP). The most common method in MOP is the point estimate weighted-sums approach [39], which characterizes the noninferior solution in terms of the optimal solution of a composite objective function. Each objective is multiplied by a strictly positive scalar weight and the weighted objectives sum into the composite objective function. It is natural to take the weights of \( f_{\text{int},i}, i=1,2,\ldots,m \) equally, so all of them are taken to be unity. Then the composite objective function is just \( \sigma \), as given by Eq. (28). Hence we may reduce \( \xi_i \) by minimizing \( \sigma \). From Eq. (24), this can be done by minimizing \( p(-w_c) \).

Therefore, \( p(-w_c) \) is relevant to the inclination angles of contact forces, and a small \( p(-w_c) \) benefits the grasp stability.

### 5.2 Constraints on the Grasp Configuration

First, the contact points should be restricted within some smooth pieces of the object surface. If a contact is located at a singular point, then \( n_i = c_i = 0 \). We can be formulated. We denote this constraint by \( R_i \in R_c, i=1,2,\ldots,m \) where points in region \( R_c \) are nonsingular.

Second, \( \dim \mathcal{W} = 6 \) is necessary to force-closure (Fig. 5). Herein we propose two necessary conditions for \( \dim \mathcal{W} = 6 \), which are directly related to the grasp configuration. If these conditions are fulfilled, then \( \dim \mathcal{W} = 6 \) in general.

**Proposition 6.** \( \dim \mathcal{W} = 6 \) only if the following conditions are both satisfied: (1) at least three contact points are noncollinear; (2) at least two unit inward normals are different.

**Proof.** If condition (1) is not satisfied, i.e., all the contact points are collinear, then rank \( G = 5 \) [40]. As \( \mathcal{W} \) lies in the range of \( G \), \( \dim \mathcal{W} \leq \text{rank } G < 6 \).

Suppose that condition (2) is not satisfied, i.e., \( n_i = n_i \) for all \( i = 2,3,\ldots,m \). Then \( o_i = o_i = 0_i = 0_i \) for \( i=2,3,\ldots,m \). Applying elementary column operations to the matrix \( T \), we obtain

\[
T = [0 w_{12} - w_{11} \cdots w_{in} - w_{in} w_{11} - w_{in} \cdots w_{in} - w_{in}]
\]

where

\[
T = [0 G_1 (s_2 - s_1) \cdots G_1 (s_m - s_1) G_2 - G_1 s_1 \cdots (G_m - G_1) s_1]
\]

Apparent, rank \( T = \text{rank } \bar{T} \). Partition \( \bar{T} \) as
contacts such that their positions fixtured differently for removing the surplus gold over the specific cone.

The initial grasp is not force-closure since rank \( r_{1} = 6 \). Second, four fixels are located collinearly (Fig. 7(b)): 

\[
\begin{align*}
\mathbf{r}_1 &= [0 - 40 \, 15 \, 
\sqrt{3}]^T, & \mathbf{r}_2 &= [15 \, 0 \, 10 \, \sqrt{3}]^T, \\
\mathbf{r}_3 &= [0 \, 15 \, \sqrt{3}/2 \, 15 \, \sqrt{3}]^T, & \mathbf{r}_4 &= [0 \, 40 \, 15 \, \sqrt{3}]^T,
\end{align*}
\]

Similarly, \( \dim \mathcal{W} = \text{rank } \mathbf{T} = 5, \) \( \mathbf{T}^T \mathbf{w}_2 - \mathbf{w}_1 = 0. \) The required CPU time is 68.60 ms.

Third, the fixels are relocated (Fig. 7(c)):

\[
\begin{align*}
\mathbf{r}_1 &= [0 - 40 \, 15 \, \sqrt{3}]^T, & \mathbf{r}_2 &= [15 \, 0 \, 10 \, \sqrt{3}]^T, \\
\mathbf{r}_3 &= [15 \, \sqrt{3}/2 \, 0 \, 15 \, \sqrt{3}]^T, & \mathbf{r}_4 &= [-15 \, 0 \, 10 \, \sqrt{3}]^T,
\end{align*}
\]

Then \( \dim \mathcal{W} = \text{rank } \mathbf{T} = 6 \) and \( \mathbf{T}^T \mathbf{w}_3 - \mathbf{w}_1 = 0. \) The proposed force-closure test algorithm verifies that only (d) is force-closure. Neither (a) nor (b) is force-closure, since \( \dim \mathcal{W} < 6 \) in them. If the dimension is ignored, the test for (a) will fall into endless computation and (b) will be mistaken for force-closure, because \( \mathbf{0} \in \text{int } \mathcal{W} \). (c) fails due to \( \mathbf{0} \notin \text{int } \mathcal{W} \). The convex hulls \( \mathcal{W} \) for (a)–(d) are represented in Figs. 4(a)–4(d).

Example 2. The object to be grasped is a jar, whose surface consists of a surface of revolution \( \mathcal{S} \) and a plane \( \mathcal{P} \), as shown in Fig. 8. They can be expressed in parametric form:

\[
\begin{align*}
\mathbf{r}_{x} &= (10 \cos (\pi v/30) + 20) \cos \phi, & \mathbf{r}_{y} &= p \cos \theta, \\
\mathbf{r}_{z} &= (10 \cos (\pi v/30) + 20) \sin \phi, & \mathbf{r}_{w} &= p \cos \theta, \\
\mathbf{r}_{v} &= v
\end{align*}
\]

where \(-20 \leq v \leq 40, 0 < \phi < 2 \pi, 0 \leq p \leq 15, 0 \leq \theta \leq 2 \pi.\) Let us determine an optimal force-closure grasp on the jar with three frictional point contacts. Contacts 1 and 2 are located on \( \mathcal{S} \) and contact 3 is located on \( \mathcal{P} \). The grasp can be specified by \( \mathbf{u} = [v_1 \, v_2 \, v_3 \, v_4] \) and the constraint on \( \mathbf{u} \) is given by \([-10 \, e -10 \, \pi +e \, 0,0] \leq \mathbf{u} \leq [30 \, \pi -e \, 30 \, \pi +e \, 10,2, \pi] \) where \( e = 0.01. \) The initial grasp is not force-closure since rank \( \mathbf{G} = 5, \) while the
rank of any grasp matrix subject to the constraint is 6. The iterative procedure of Eq. (37) is described in Fig. 9. A force-closure grasp is obtained in five iterations with the CPU time of 12.28 s (u_\text{avg}=\{1.9705 0.0796 \pi, 1.9457 1.0291 \pi , 4.2551 0.0813 \pi \}) for which \( p(-w_i) = 0.4889 \). After the 15th iteration, \( u_{15} \equiv \{5.9022 0.0160 \pi , 5.8499 1.0238 \pi , 0.0008 0.0990 \pi \} \) and \( p(-w_i) = 0.0230 \). Thus we obtain an optimal grasp as depicted in Fig. 8. The CPU time is 38.60 s.

7 Conclusion

1. The convex hulls of primitive wrenches are classified into four categories by their dimensions and relative positions to the origin of the wrench space (Fig. 4). It is shown that a grasp is force-closure if and only if the convex hull is 6D and the origin is its relative interior point.
2. We point out the importance of the dimension of the convex hull. Disregarding the dimension, the original ray-shooting approach may make mistakes in force-closure test [4]. To avoid such mistakes, we supplement the condition for the convex hull being 6D; that is, the grasp matrix is full row rank and a certain linear system is consistent.
3. Whether the origin is a relative interior point of the convex hull is determined by the simplified ray-shooting approach. Its geometric meaning is illustrated clearly in Fig. 6. The consistency of a certain linear system guarantees that the LP formulation in this approach always has an optimal solution.
4. As a whole, an exact and efficient force-closure test algorithm is developed.

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Nomenclature

- \( m \): number of contacts
- \( r_i \): position vector of contact \( i (i = 1, 2, \ldots, m) \)
- \( n_i, o_i, t_i \): unit inward normal and two tangent vectors at contact \( i \), \( n_i = o_i \times t_i \)
- \( \mu \): Coulomb friction coefficient at contact
- \( f_i \): contact force at contact \( i \)
- \( f_{\text{int}}, f_{\text{ext}} \): force components along \( n_i, o_i, t_i \)
- \( G_i \): grasp matrix at contact \( i \)
- \( w_i \): wrench produced by \( f_i \) on the object
- \( f \): total contact force, \( f = [f_1^T f_2^T \cdots f_m^T]^T \in R^{3m} \)
- \( G \): total grasp matrix, \( G = [G_1 G_2 \cdots G_m] \in R^{6 \times 3m} \)
- \( w_{\text{ext}} \): external wrench on the object
- \( n \): number of side edges for linearizing the friction cone
- \( s_j \): the \( j \)th edge vector of the polyhedral cone \( j = 1, 2, \ldots, n \)
- \( w \): primitive wrench
- \( \mathcal{W} \): convex hull of the primitive wrenches
- \( w_c \): centroid of the primitive wrenches
- \( T \): translate of \( \mathcal{W} \) by \( w \)
- \( T = [w_1+\cdots+w_n, \cdots w_{nm}+w_n] \in R^{6 \times mn} \)
- \( T' \): polar set of \( T \)
- \( p \): support function of \( T' \)
- \( p(-w_i) \): value of \( p \) with respect to \( -w_i \)
- \( \mathbf{0} \): origin of a space
- \( R_{\mathbf{0}} \): ray from \( w_i \) to \( \mathbf{0} \)
- \( \mathbf{aff}(\cdot) \): affine hull of a set
- \( \dim(\cdot) \): dimension of a set, i.e., the dimension of the affine hull of the set
- \( r_i(\cdot) \): relative interior of a set, i.e., the interior of the set in its affine hull
- \( c(l)(\cdot) \): closure of a set
- \( r(b)(\cdot) \): relative boundary of a set
- \( i(n)(\cdot) \): interior of a set
- \( c(v)(\cdot) \): convex hull of a set

References

Grippers for Positioning and Identifying Objects,” IEEE Trans. Syst. Man
Force Distribution and Load Capacity of Grasps and Fixtures,” ASME J.
rem With Applications to Multi-Fingered Grasping,” Proceedings of the 22nd
669–679.
That Result in High Quality and Robust Configurations,” J. Rob. Syst.,