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On some weaknesses existing in optimal grasp planning

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Abstract

Optimal grasp planning has been investigated for over two decades. Still some important weaknesses in the previous work are worthy of notice: (i) The kinematic structure and the geometric configuration of a robot hand were ignored. Fingers were assumed to be capable of contacting an object anywhere. This is unrealistic. (ii) The grasp quality criterion was general and often did not match the task requirement. (iii) The criterion depends on the choice of unit and coordinate frame and lacks a clear physical meaning. This paper tries to remedy them. First, a general technique is proposed to find all feasible grasps on an object conforming to the robot hand. Next, for a specified external wrench or an external wrench set of a certain task, the maximum equilibrating contact force is adopted as the grasp quality criterion. Having an evident meaning, it is independent of the choice of unit and coordinate frame. Finally, an algorithm is presented for seeking the globally optimal grasp for which the value of the criterion is minimal among the feasible ones.

Keywords: Optimal grasp planning (OGP); Feasibility; Optimality criterion; Multifingered grasping; Robot hand

1. Introduction

Optimal grasp planning (OGP) is a topic of finding the optimal contact positions on an object such that a robot hand can manipulate it with high performance quality. This topic has been investigated with great enthusiasm since the pioneer work of Salisbury and Roth [1]. A number of significant results have been obtained, mainly in the following three aspects:

1. *Force-closure conditions and tests*: The force-closure property, including form-closure as a frictionless case, is prerequisite to stable grasping. It means the capability of a grasp to restrain any motion and to equilibrate any external wrench on the grasped object. It is well-known that a grasp is force-closure if and only if the primitive wrenches positively span the wrench space [1], or the origin of the wrench space lies strictly in the convex hull of the primitive wrenches [2]. Murray et al. [3] deduced a condition in the contact force

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space; that is, the grasp matrix is surjective and there is a strictly internal force. By the duality between the infinitesimal motion and the wrench, Zheng and Qian [4] generalized the method of form-closure analysis [5–7] to force-closure. Based on these conditions, testing algorithms were developed [8–12].
2. *Grasp quality evaluation*: To compare the goodness of force-closure grasps, the magnitude of the maximum external wrench in the worst direction that can be equilibrated with unit contact forces was suggested to be a grasp quality criterion [13,14]. Borst et al. [15], Zhu and Wang [16], Liu et al. [17], and Zheng and Qian [18] proposed different computational methods. Noticing that a wrench consists of a force and a moment, which has different dimensions. Mirtich and Canny [19] treated them separately in assessing the grasp quality. By doing this, the ambiguity in the physical meaning of the wrench magnitude was eliminated.
3. *Planning techniques*: Formerly, study focused on synthesizing force-closure grasps on simple objects with limited contacts. On polygonal objects, Nguyen [20] computed independent regions for two frictional or four frictionless point contacts. Markenscoff and Papadimitriou [21] calculated the optimal grip. Park and Starr [22] synthesized a three-finger grasp, while Tung and Kak [23] constructed a two-finger one. On irregular 2D and 3D objects, Chen and Burdick [24] considered two-finger antipodal point grasps, and Li et al. [25] developed a geometrical algorithm for computing three-finger force-closure grasps. Ponce et al. [26] promoted Nguyen's idea [20] to four-finger force-closure grasps on polyhedral objects. In recent years, the planning scope becomes more general: the contact number is no more limited and the object surface requires piecewise smooth only. Liu [27] computed 2D n-finger grasps, while Ding et al. [28] considered 3D n-finger grasps where k fingers are located in advance. Following the gradient flows of the quality criterion [13,14], Zhu and Wang [16], and Liu et al. [17] synthesized optimal grasps on 3D objects with piecewise smooth surface. Liu et al. [29] sought force-closure grasps on 3D objects represented by discrete points. In addition, some algorithms for fixture design can be applied to grasp planning as well [30].

In the OGP research to date, however, there are some hidden weaknesses:

1. It was assumed that fingers could make contact with an object at any locations. Actually, the working space of a finger is restricted. Whether and where it contacts an object are relevant to its kinematic structure and geometric configuration as well as the pose of the hand relative to the object. Therefore contacts cannot be located so independently and freely. Even a dexterous hand with multiple degrees of freedom can hardly fulfill this assumption.
2. An optimal grasp was required to equilibrate the external wrench in all directions as efficiently as possible. But the external wrench on an object in a specific task is specified in certain directions, and some directions need more attention. Thus a task-oriented grasp quality criterion should be used in order that the optimized grasp owns the best performance quality indeed, such as the one presented in [31,32]. Shimoga [32] also described several other quality criteria, which in certain cases might be more important.
3. The grasp quality criterion [13,14] depends on the choice of unit and coordinate frame. This problem arises because the force and moment components of a wrench have different units and the latter is not invariant to translations of the origin. Consequently, the calculated optimal grasps are unit and frame dependent.

This paper aims to dispose of these problems. Firstly, by means of the L_2 distance between compact sets, we put forward a general method of seeking grasps on an object conforming to a robot hand. This method can be applied to various 3D objects and robot hands. Rather than a single feasible grasp, all of them are figured out for finding the best one. Next, given a specific manipulation task, in which the dynamic external wrench is formulated or the set of external wrenches is specified, we utilize the largest contact force required for equilibrium to assess the grasp performance quality. This criterion is irrelevant to the choice of unit and coordinate frame. By extending our previous work on dynamic force distribution [18], an efficient method for computing it is derived. At last, among the feasible grasps the global optimum for which the value of the criterion is minimal is searched out by an algorithm. Through these efforts, OGP is rid of the foregoing weaknesses eventually.

2. Feasible grasps on an object for a robot hand

2.1. Representation of a robot hand relative to an object

We first establish a mathematical model of grasping. For detailed knowledge about this part, one may refer to [3]. Let O denote an object fixed with a coordinate frame $\mathscr{F}_O$, in which O is represented as a compact set. Assume that an m-finger robot hand fixed with a coordinate frame $\mathscr{F}_H$ is used to grasp O. Let $\boldsymbol{p}_{OH} \in \mathbb{R}^3$ be the position vector of the origin of $\mathscr{F}_H$ in $\mathscr{F}_O$ and $\boldsymbol{R}_{OH} \in SO(3)$ the orientation of $\mathscr{F}_H$ relative to $\mathscr{F}_O$. The values of $\boldsymbol{p}_{OH}$ and $\boldsymbol{R}_{OH}$ are determined by the robot arm that the hand is connected with.

Assume that each finger contacts the object at the fingertip. Let $\mathscr{F}_i$ for $i = 1, 2, \ldots, m$ be a coordinate frame attached to fingertip i, so that fingertip i can be expressed in $\mathscr{F}_i$ by a compact convex set F_i. The motion of a finger has to conform to its kinematic structure. Let $\boldsymbol{\beta}_i$ be a vector comprising several variables that result in the motion of F_i, and $\underline{\boldsymbol{\beta}}_i$ and $\bar{\boldsymbol{\beta}}_i$ the lower and upper bounds of $\boldsymbol{\beta}_i$. Then F_i in $\mathscr{F}_H$ can be written as a function of $\boldsymbol{\beta}_i$, i.e.,

$$F_{iH} = \boldsymbol{p}_i(\boldsymbol{\beta}_i) + \boldsymbol{R}_i(\boldsymbol{\beta}_i)F_i \tag{1}$$

where $\boldsymbol{p}_i(\boldsymbol{\beta}_i) \in \mathbb{R}^3$ and $\boldsymbol{R}_i(\boldsymbol{\beta}_i) \in SO(3)$. Consequently, F_i in $\mathscr{F}_O$ is given by

$$F_{iO} = \boldsymbol{p}_{OH} + \boldsymbol{R}_{OH}F_{iH} = \boldsymbol{p}_{OH} + \boldsymbol{R}_{OH}\boldsymbol{p}_i(\boldsymbol{\beta}_i) + \boldsymbol{R}_{OH}\boldsymbol{R}_i(\boldsymbol{\beta}_i)F_i \tag{2}$$

Eq. (2) implies that the position of F_i relative to O is determined by $\boldsymbol{p}_{OH}, \boldsymbol{R}_{OH}$, and $\boldsymbol{\beta}_i$. Thus the values of $\boldsymbol{p}_{OH}, \boldsymbol{R}_{OH}$, and $\boldsymbol{\beta}_i, i = 1, 2, \ldots, m$ determine the pose of the robot hand.

Moreover, motions of fingers may be restrained by each other. Such constraints can be generally formulated as

$$g(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_m) = 0 \tag{3}$$

There are other forms of grasps, in which not only the fingertips but also the palm and/or the phalanges make contact with the object, such as power grasp. Fixed on the robot hand, the palm can be expressed in $\mathscr{F}_H$ by a compact set, which is transformed into $\mathscr{F}_O$ by $\boldsymbol{p}_{OH}$ and $\boldsymbol{R}_{OH}$. Each phalange can be represented as a set in a coordinate frame attached to it, and the set can also be transformed into $\mathscr{F}_O$. Only one more thing needs to be noted, that is, a phalange and a fingertip share some components of $\boldsymbol{\beta}_i$. Therefore the mathematical models of other grasp forms are essentially identical with the one of fingertip grasp. For simplicity, hereinafter we follow the foregoing assumption, but the proposed method is generally applicable.

2.2. Conformity of a grasp on an object to a robot hand

A grasp on an object is a set of points on the object surface, where the robot hand contacts the object with its fingers. Let $\boldsymbol{r}_i \in \mathbb{R}^3, i = 1, 2, \ldots, m$ be the position vectors of contacts in $\mathscr{F}_O$. Then the grasp is denoted by

$$G = \{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_m\} \tag{4}$$

Because of the constraints (2) and (3), the contact points cannot be arbitrarily selected on the object surface. They depend on the structure of the robot hand and its pose relative to the object. In what follows, we shall introduce the mathematical formulations of *contact* and *conformity* of a grasp.

Proposition 1. *The finger F_{iO} can contact the object O if and only if there exist $\boldsymbol{p}_{OH} \in \mathbb{R}^3, \boldsymbol{R}_{OH} \in SO(3)$, and $\boldsymbol{\beta}_i \in [\underline{\boldsymbol{\beta}}_i, \bar{\boldsymbol{\beta}}_i]$ such that $F_{iO} \cap O \neq \emptyset$ and $\mathrm{int}\,F_{iO} \cap \mathrm{int}\,O = \emptyset$, where $\mathrm{int}(\cdot)$ denotes the interior of a set.*

Proposition 2. *The robot hand can grasp the object O with all the fingers if and only if $F_{iO}, i = 1, 2, \ldots, m$ can contact O subject to (3).*

Definition 1. A grasp on an object is said to be feasible for a robot hand or conform to it if the hand can contact the object by the grasp with all its fingers.

Only feasible grasps can be put into practice. Their existence is the prerequisite for OGP.

3. General method of seeking feasible grasps

In this section, we first introduce the L_2 distance between two sets, which can be used as an ideal measure of contact. Then conditions and an algorithm for seeking feasible grasps are proposed.

3.1. The L_2 distance between nonempty compact sets

In convex analysis and computational geometry, the distance between sets is often defined as the minimum distance between their elements [33–35]. This traditional distance cannot distinguish penetration from contact between sets because its value is zero in both cases. Although the growth distance [36] and the pseudodistance [37] can, their values are usually not equal to the exact distance in the physical sense. Thus we prefer the L_2 distance, which does not have these troubles. In our previous work [18], the L_2 distance between the origin and a nonempty compact convex set together with its computation was addressed. For use in this paper, herein we extend its scope to two nonempty compact sets. Some extensions are inspired by [34,35].

Definition 2. Let S_1 and S_2 be compact convex sets with nonempty interiors and B_0 the unit ball in terms of the L_2 metric centered at the origin $\mathbf{0}$. The L_2 distance between S_1 and S_2 is defined by

$$\rho(S_1, S_2) = \begin{cases} \min\limits_{\lambda B_0 \cap (S_2 - S_1) \neq \emptyset,\, \lambda \geqslant 0} \lambda, & \text{if } S_1 \cap S_2 = \emptyset \\ \min\limits_{\lambda B_0 \subset S_2 - S_1,\, \lambda \leqslant 0} \lambda, & \text{if } S_1 \cap S_2 \neq \emptyset \end{cases} \tag{5}$$

where $S_2 - S_1 = \{\mathbf{x}_2 - \mathbf{x}_1 | \mathbf{x}_1 \in S_1, \mathbf{x}_2 \in S_2\}$ is the Minkowski difference between S_1 and S_2 (Fig. 1).

Theorem 1. *Let $S = S_2 - S_1$. Then the following statements are true*:

1. *S is a compact convex set with nonempty interior.*
2. *$\mathbf{0} \in S$ if and only if $S_1 \cap S_2 \neq \emptyset$.*
3. *$\mathbf{0} \in \operatorname{int} S$ if and only if $\operatorname{int} S_1 \cap \operatorname{int} S_2 \neq \emptyset$.*

Proof. The proof of parts (1) and (2) is straightforward. Part (3) follows from part (2) and $\operatorname{int} S = \operatorname{int} S_2 - \operatorname{int} S_1$. □

According to Theorem 1, parts (1) and (2), $\rho(S_1, S_2)$ can be rewritten as

$$\rho(S_1, S_2) = \rho(\mathbf{0}, S) = \begin{cases} \min\limits_{\lambda B_0 \cap S \neq \emptyset, \lambda \geqslant 0} \lambda, & \text{if } \mathbf{0} \notin S \\ \min\limits_{\lambda B_0 \subset S, \lambda \leqslant 0} \lambda, & \text{if } \mathbf{0} \in S \end{cases} \tag{6}$$

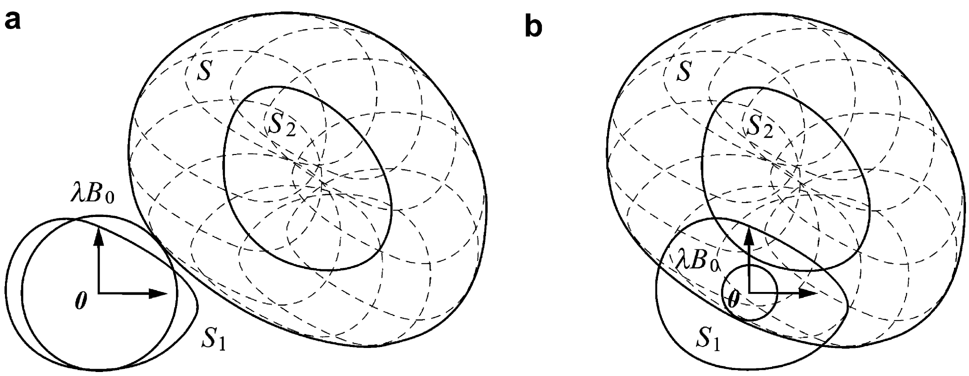

Fig. 1. Illustrating the definition of the L_2 distance.

Theorem 2. *The following statements are true*:

1. $\rho(S_1, S_2) > 0$ *if and only if* $S_1 \cap S_2 = \emptyset$.
2. $\rho(S_1, S_2) = 0$ *if and only if* $S_1 \cap S_2 \neq \emptyset$ *and* $\mathrm{int}\, S_1 \cap \mathrm{int}\, S_2 = \emptyset$.
3. $\rho(S_1, S_2) < 0$ *if and only if* $\mathrm{int}\, S_1 \cap \mathrm{int}\, S_2 \neq \emptyset$.

Proof. It follows straightforwardly from (6) and Theorem 1. □

In words, Theorem 2(1) implies that S_1 and S_2 are separated, and $\rho(S_1, S_2)$ is the radius of the largest open ball centered at $\mathbf{0}$ without intersecting S. Part (2) means that S_1 just contacts S_2. Part (3) indicates that S_1 and S_2 are penetrated, and $-\rho(S_1, S_2)$ is the radius of the largest ball centered at $\mathbf{0}$ contained in S.

Similarly to the traditional distance [34], the L_2 distance can be expanded into a rich family of nonconvex sets which are the unions of compact convex sets and their spherical extensions. This will facilitate the application of the L_2 distance to the objects comprising some convex parts (see Examples 1 and 2) and having round corners (see Example 2). Suppose that S_1 and S_2 are each the union of a finite number of compact convex sets, i.e.,

$$S_1 = \bigcup_{k_1 \in I_1} S_{k_1}, \quad S_2 = \bigcup_{k_2 \in I_2} S_{k_2} \tag{7}$$

where S_{k_1} and S_{k_2} are convex sets, and I_1 and I_2 are index sets. Then

$$\rho(S_1, S_2) = \min_{k_1 \in I_1, k_2 \in I_2} \rho(S_{k_1}, S_{k_2}) \tag{8}$$

The spherical extensions of S_1 and S_2 are defined by

$$S_1^{r_1} = S_1 + r_1 B_0, \quad S_2^{r_2} = S_2 + r_2 B_0 \tag{9}$$

It is easy to know that

$$\rho(S_1^{r_1}, S_2^{r_2}) = \rho(S_1, S_2) - r_1 - r_2. \tag{10}$$

More generally, let

$$S_1 = \bigcup_{k_1 \in I_1} S_{k_1}^{r_{k_1}}, \quad S_2 = \bigcup_{k_2 \in I_2} S_{k_2}^{r_{k_2}} \tag{11}$$

where $S_{k_1}^{r_{k_1}}$ and $S_{k_2}^{r_{k_2}}$ are the spherical extensions of S_{k_1} and S_{k_2}, respectively. Then

$$\rho(S_1, S_2) = \min_{k_1 \in I_1, k_2 \in I_2} \{\rho(S_{k_1}, S_{k_2}) - r_{k_1} - r_{k_2}\} \tag{12}$$

Let $p_S(\mathbf{z}) = \sup_{x \in S} \mathbf{z}^{\mathrm{T}} \mathbf{x}$ denote the support function of S [33], where $\mathbf{z}$ is a vector. From (6) and the formula for computing $\rho(\mathbf{0}, S)$ derived in [18], we directly obtain

Theorem 3. $\rho(S_1, S_2) = \rho(\mathbf{0}, S) = -\min_{\|\mathbf{z}\|=1} p_S(\mathbf{z})$.

The following theorem is quite useful for computing $p_S(\mathbf{z})$.

Theorem 4. *Let A and B be two compact sets. Then the following statements are true*:

1. $p_{A \pm B}(\mathbf{z}) = p_A(\mathbf{z}) + p_B(\pm \mathbf{z})$.
2. $p_{\mathrm{conv}\, A}(\mathbf{z}) = p_A(\mathbf{z})$, *where* $\mathrm{conv}(\cdot)$ *denotes the convex hull of a set.*
3. $p_{A \cup B}(\mathbf{z}) = \max\{p_A(\mathbf{z}), p_B(\mathbf{z})\}$.
4. $p_{\mathbf{R}(A)}(\mathbf{z}) = p_A(\mathbf{R}^{\mathrm{T}} \mathbf{z})$, *where* $\mathbf{R}$ *is a matrix denoting a linear mapping.*
5. *Suppose that A and B are subsets of* R^{n_1} *and* R^{n_2}, *respectively. Let* $\mathbf{z} = [\mathbf{z}_1^{\mathrm{T}}\ \mathbf{z}_2^{\mathrm{T}}]^{\mathrm{T}} \in \mathrm{R}^{n_1+n_2}$, *where* $\mathbf{z}_1 \in \mathbb{R}^{n_1}$ *and* $\mathbf{z}_2 \in \mathbb{R}^{n_2}$. *Then* $p_{A \times B}(\mathbf{z}) = p_A(\mathbf{z}_1) + p_B(\mathbf{z}_2)$.

Proof. The proof is straightforward. Detailed explanation of parts 1–4 can be found in [34]. □

From Theorems 3 and 4(1), $\rho(S_1, S_2)$ can be computed by

$$\rho(S_1, S_2) = -\min_{\|z\|=1}\{p_{S_2}(z) + p_{S_1}(-z)\} \tag{13}$$

Eq. (13) is illustrated in Fig. 2. The functions $p_{S_1}(-z)$ and $p_{S_2}(z)$ together with the vector z determine a pair of parallel hyperplanes H_1 and H_2 that support S_1 and S_2, respectively, i.e.,

$$H_1 = \{x| - z^{\mathrm{T}}x = p_{S_1}(-z)\} \quad \text{and} \quad H_2 = \{x|z^{\mathrm{T}}x = p_{S_2}(z)\} \tag{14}$$

The value of $p_{S_1}(-z)$ (resp. $p_{S_2}(z)$ is the distance from $\mathbf{0}$ to H_1 (resp. H_2) along $-z$ (resp. z). The value of $p_S(z)$ is the distance from H_1 to H_2 along z. Thus $-\rho(S_1, S_2)$ is the minimum value of the directional distances between all such pairs of supporting hyperplanes given by (14).

Theorem 5. *Suppose that $\hat{z}$ is the vector such that $\rho(S_1, S_2) = \rho(\mathbf{0}, S) = -p_S(\hat{z})$. Let $\hat{S}_1$ (resp. $\hat{S}_2$) be the set of all points $x \in S_1$ (resp. $x \in S_2$) such that $p_{S_1}(-\hat{z}) = -\hat{z}^T x$ (resp. $p_{S_2}(\hat{z}) = \hat{z}^{\mathrm{T}} x$). If $\rho(S_1, S_2) = 0$, then $\hat{S}_1 \cap \hat{S}_2$ consists of the contact points of S_1 and S_2, and $\hat{z}$ is the normal at contacts inward to S_1 and outward to S_2.*

Compared with existing literature on the distance between objects, especially the outstanding achievements of Gilbert and his colleagues [34–36], the formulas regarding $\rho(S_1, S_2)$ in this subsection have a certain distinction in computing the penetration, which will be used in the algorithm for seeking feasible grasps. In addition, Theorems 4(5) and 5 are new results of convex analysis.

3.2. Feasibility condition and searching algorithm

According to Theorem 2, the L_2 distance can be used to detect penetration, contact, or separation between the fingers and the object. If the object O is convex, then directly from Theorem 3 we may compute the L_2 distance between the finger F_{iO} and O by

$$\rho(F_{iO}, O) = -\min_{\|z\|=1}\{p_O(z) + p_{F_{iO}}(-z)\} \tag{15}$$

If O is composed of some convex parts, we compute $\rho(F_{iO}, O)$ by (8). With the help of Theorem 4, the explicit expressions of $p_O(z)$ and $p_{F_{iO}}(-z)$ can be derived from the formulations of O and F_{iO}, respectively. Consequently, $\rho(F_{iO}, O)$ can be computed by solving the optimization problem (15). Let $z = [\cos\alpha_1 \cos\alpha_2 \quad \cos\alpha_1 \sin\alpha_2 \quad \sin\alpha_1]^{\mathrm{T}}$, and then the problem (15) can be rewritten as

$$\rho(F_{iO}, O) = -\min_{-\pi/2 \leqslant \alpha_1 \leqslant \pi/2,\, 0 \leqslant \alpha_2 \leqslant 2\pi}\{p_O(z) + p_{F_{iO}}(-z)\} \tag{16}$$

From Propositions 1 and 2 together with Theorems 2 and 5, we have

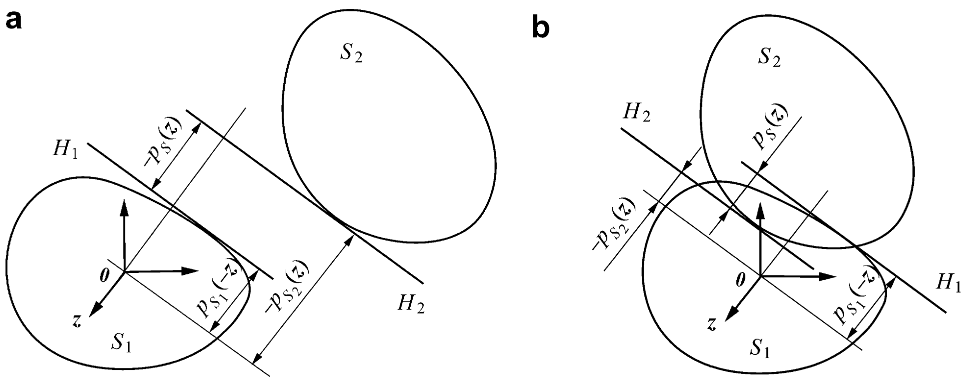

Fig. 2. Illustrating the computation of the L_2 distance.

Proposition 3. *The finger F_{iO} can make contact with the object O if and only if there exist $\boldsymbol{p}_{OH} \in \mathbb{R}^3, \boldsymbol{R}_{OH} \in SO(3)$, and $\boldsymbol{\beta}_i \in [\underline{\boldsymbol{\beta}}_i, \bar{\boldsymbol{\beta}}_i]$ such that $\rho(F_{iO}, O) = 0$. Furthermore, the unit vector $\hat{\boldsymbol{z}}$, for which $\rho(F_{iO}, O) = p_O(\hat{\boldsymbol{z}}) + p_{F_{iO}}(-\hat{\boldsymbol{z}}) = 0$, determines the contact points and the normal therein.*

Proposition 4. *The values of $\boldsymbol{p}_{OH} \in \mathbb{R}^3, \boldsymbol{R}_{OH} \in SO(3)$, and $\boldsymbol{\beta}_i \in [\underline{\boldsymbol{\beta}}_i, \bar{\boldsymbol{\beta}}_i], i = 1, 2, \ldots, m$, for which $\rho(F_{iO}, O) = 0$ and $g(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_m) = 0$, determine a grasp on the object O feasible for the robot hand.*

Algorithm 1 (*Searching for feasible grasps*)

Step 1. Set the initial values of $\boldsymbol{p}_{OH}$ and $\boldsymbol{R}_{OH}$.
Step 2. Search the domain $[\underline{\boldsymbol{\beta}}_i, \bar{\boldsymbol{\beta}}_i]$ for $\boldsymbol{\beta}_i$ such that $\rho(F_{iO}, O) = 0$. Methods for solving equations, such as the bisection method, are available for this purpose. The solution might be not unique; then denote them by $\boldsymbol{\beta}_{i,j_i}$, $j_i = 1, 2, \ldots, J_i$. If the solution does not exist for some i, then go to Step 5.
Step 3. Compute the values of $g(\boldsymbol{\beta}_{1,j_1}, \boldsymbol{\beta}_{2,j_2}, \ldots, \boldsymbol{\beta}_{m,j_m})$ for each $j_i = 1, 2, \ldots, J_i$ and $i = 1, 2, \ldots, m$. If none of them are zero, then go to Step 5.
Step 4. Calculate the contact points w.r.t. $\boldsymbol{\beta}_{1,j_1}, \boldsymbol{\beta}_{2,j_2}, \ldots, \boldsymbol{\beta}_{m,j_m}$, for which $g(\boldsymbol{\beta}_{1,j_1}, \boldsymbol{\beta}_{2,j_2}, \ldots, \boldsymbol{\beta}_{m,j_m}) = 0$. Each set of such contact points gives a feasible grasp.
Step 5. To seek other feasible grasps, change $\boldsymbol{p}_{OH}$ and $\boldsymbol{R}_{OH}$, and return to Step 2; otherwise, the algorithm ends.

4. Optimal grasp planning for a specific manipulation task

4.1. Grasp quality criterion

Set a local coordinate frame at each contact point with the unit inward normal $\boldsymbol{n}_i$ and the unit tangent vectors $\boldsymbol{o}_i$ and $\boldsymbol{t}_i$. The contact force $\boldsymbol{f}_i$ $(i = 1, 2, \ldots, m)$ can be expressed in the local coordinate frame by

$$\boldsymbol{f}_i = [f_{in} \quad f_{io} \quad f_{it}]^{\mathrm{T}} \in \mathbb{R}^3 \tag{17}$$

where f_{in}, f_{io}, and f_{it} are the force components along $\boldsymbol{n}_i, \boldsymbol{o}_i$, and $\boldsymbol{t}_i$, respectively. To avoid separation and slip at contact, $\boldsymbol{f}_i$ must comply with the contact constraint:

$$f_{in} \geqslant 0 \quad \text{and} \quad \sqrt{f_{io}^2 + f_{it}^2} \leqslant \mu_i f_{in} \tag{18}$$

where μ_i is the Coulomb friction coefficient. The overall magnitude of contact forces is measured by the index

$$\sigma = \max_{i=1,2,\ldots,m} \left(\frac{f_{in}}{f_i^{\mathrm{U}}} \right) \tag{19}$$

where f_i^{U} is the upper bound of f_{in}, bounded by the material strength and/or the actuator power. To equilibrate an external wrench $\boldsymbol{w}_{\mathrm{ext}}$ on the object O, the resultant wrench $\boldsymbol{w}$ of $\boldsymbol{f}_i, i = 1, 2, \ldots, m$ must satisfy

$$\boldsymbol{w} = \sum_{i=1}^{m} \boldsymbol{G}_i \boldsymbol{f}_i = -\boldsymbol{w}_{\mathrm{ext}} \tag{20}$$

where $\boldsymbol{G}_i$ is the grasp matrix:

$$\boldsymbol{G}_i = \begin{bmatrix} \boldsymbol{n}_i & \boldsymbol{o}_i & \boldsymbol{t}_i \\ \boldsymbol{r}_i \times \boldsymbol{n}_i & \boldsymbol{r}_i \times \boldsymbol{o}_i & \boldsymbol{r}_i \times \boldsymbol{t}_i \end{bmatrix} \in \mathbb{R}^{6\times3} \tag{21}$$

There are many solutions of (20) in (18) for a given $\boldsymbol{w}_{\mathrm{ext}}$. Among them, we select the minimum contact forces, whose σ is minimal. Let $\sigma_{\min}$ denote the minimum value of σ. Then we have

$$\sigma_{\min} = \min_{\boldsymbol{f}_i, i=1,2,\ldots,m \text{ satisfy (18) and (20)}} \sigma = \min_{\boldsymbol{f}_i, i=1,2,\ldots,m \text{ satisfy (18) and (20)}} \left\{ \max_{i=1,2,\ldots,m} \left(\frac{f_{in}}{f_i^{\mathrm{U}}} \right) \right\}. \tag{22}$$

In some manipulation tasks, a specific motion of the object is given by a function of time t. Then the external wrench is determined by the Newton–Euler equation [3], which is also specified as a function of time t. In some others, how the object interacts with the environment is foreseen, and then the expected external wrench can be described by a compact wrench set [31]. In the former case, the performance quality of a grasp G is naturally assessed by the maximum (during the variation of $\boldsymbol{w}_{\text{ext}}$) of the minimum contact forces (among all possible solutions at choice) arising in the task, which can be formulated as

$$Q(G, \boldsymbol{w}_{\text{ext}}) = \max_{\boldsymbol{w}=-\boldsymbol{w}_{\text{ext}}(t_k),\, k=1,2,\ldots,K} \sigma_{\min,k} = \max_{\boldsymbol{w}=-\boldsymbol{w}_{\text{ext}}(t_k),\, k=1,2,\ldots,K} \left\{ \min_{\boldsymbol{f}_i,\, i=1,2,\ldots,m \text{ satisfy (18) and (20)}} \sigma \right\} \tag{23}$$

where t_k, $k = 1, 2, \ldots, K$ designate the sampling times during the task and $\sigma_{\min,k}$ denotes the value $\sigma_{\min}$ w.r.t. $\boldsymbol{w}_{\text{ext}}$ at time t_k. In the latter case, assume that the wrench set, denoted by W_{ext}, is polyhedral and convex. If the original external wrench set is not of this kind, a circumscribed polytope around it can be used instead. Every external wrench in the original set can be equilibrated provided those in the circumscribed polytope can be equilibrated. Let $\boldsymbol{w}_k$, $k = 1, 2, \ldots, K$ be the vertices of W_{ext}. If these vertex wrenches can be equilibrated, then every wrench in W_{ext} can be equilibrated. Similarly to the former case, the performance quality of a grasp G in this case is evaluated by the maximum of the minimum contact forces for equilibrating all external wrenches in W_{ext}. Notice that the maximum always arises at one of the vertices $\boldsymbol{w}_k, k = 1, 2, \ldots, K$. Hence the grasp quality criterion can be formulated as

$$Q(G, W_{\text{ext}}) = \max_{\boldsymbol{w}=-\boldsymbol{w}_k,\, k=1,2,\ldots,K} \sigma_{\min,k} = \max_{\boldsymbol{w}=-\boldsymbol{w}_k,\, k=1,2,\ldots,K} \left\{ \min_{\boldsymbol{f}_i,\, i=1,2,\ldots,m \text{ satisfy (18) and (20)}} \sigma \right\} \tag{24}$$

where $\sigma_{\min,k}$ denotes the value $\sigma_{\min}$ w.r.t. $\boldsymbol{w}_k$. In addition, if the external wrench at some k cannot be equilibrated, then let

$$Q(G, \boldsymbol{w}_{\text{ext}}) = -1 \quad \text{or} \quad Q(G, W_{\text{ext}}) = -1 \tag{25}$$

If all the external wrenches can be equilibrated, then $Q(G, \boldsymbol{w}_{\text{ext}})$ or $Q(G, W_{\text{ext}})$ is positive, and a smaller value indicates better grasp quality. On the other hand, a value greater than unity means that at least one contact force exceeds its upper bound. This should be avoided.

4.2. Computation of the criterion

From the above, the performance quality of a grasp w.r.t. a manipulation task can be evaluated by its efficiency of equilibrating K external wrenches. Hereinafter, $Q(G, \boldsymbol{w}_{\text{ext}})$ and $Q(G, W_{\text{ext}})$ are abbreviated to $Q(G)$, and $\boldsymbol{w}_{\text{ext}}$ at time t_k is also denoted by $\boldsymbol{w}_k$. Then the key to computing $Q(G)$ is calculating $\sigma_{\min,k}$ w.r.t. $\boldsymbol{w}_k$. This can be solved by the method of distributing contact forces [18].

Substitute the friction cone by a polyhedral convex cone with edges

$$\boldsymbol{s}_{i,h} = f_i^{\text{U}}[1 \quad \mu_i \cos(2h\pi/l) \quad \mu_i \sin(2h\pi/l)]^{\text{T}}, \quad h = 1, 2, \ldots, l \tag{26}$$

Then $\boldsymbol{f}_i$ satisfying (18) can be represented by

$$\boldsymbol{f}_i = \sum_{h=1}^{l} \lambda_{i,h} \boldsymbol{s}_{i,h}, \quad \lambda_{i,h} \geqslant 0 \qquad \text{for } h = 1, 2, \ldots, l \tag{27}$$

Combining (17), (26), and (27) leads to

$$\sigma = \max_{i=1,2,\ldots,m} \left(\sum_{h=1}^{l} \lambda_{i,h} \right) \tag{28}$$

Let $\boldsymbol{w}_{i,h} = \boldsymbol{G}_i \boldsymbol{s}_{i,h}$ for $h = 1, 2, \ldots, l$ and $i = 1, 2, \ldots, m$, $\mathscr{W}_i$ the set of $\boldsymbol{w}_{i,1}, \boldsymbol{w}_{i,2}, \ldots, \boldsymbol{w}_{i,l}$ together with the origin $\mathbf{0}$ of the wrench space, and $\mathscr{W}$ the Minkowski sum of $\mathscr{W}_i$, $i = 1, 2, \ldots, m$ except $\mathbf{0}$:

$$\mathscr{W}_i = \{\boldsymbol{w}_{i,h} | h = 1, 2, \ldots, l\} \cup \{\mathbf{0}\} \tag{29}$$

$$\mathscr{W} = \left(\sum_{i=1}^{m} \mathscr{W}_i\right) \setminus \mathbf{0} = \{\boldsymbol{w}_j | j = 1, 2, \ldots, n\} \tag{30}$$

where $n = (l+1)^m - 1$. From [18] we first see that the overall magnitude $\sigma_{\min,k}$ of the minimum contact forces w.r.t. $\boldsymbol{w}_k$ is equal to the optimal objective value of the linear programming problem:

$$\begin{cases} \text{Maximize} \ \ -\boldsymbol{w}_k^{\mathrm{T}}\boldsymbol{z} \\ \text{subject to } \boldsymbol{z}^{\mathrm{T}}\boldsymbol{w}_j \leqslant 1, \quad j = 1, 2, \ldots, n \end{cases} \tag{31}$$

Here, $\boldsymbol{z} \in \mathbb{R}^6$. Furthermore, let $\boldsymbol{z}_k$ be the optimal solution of (31) and $\boldsymbol{P}_k$ the matrix whose columns are $\boldsymbol{w}_j, j = 1, 2, \ldots, n$ satisfying $\boldsymbol{z}_k^{\mathrm{T}}\boldsymbol{w}_j = 1$. Then $\boldsymbol{\lambda} = [\lambda_{1,1} \quad \lambda_{1,2} \quad \cdots \quad \lambda_{m,l}]^{\mathrm{T}}$ corresponding to $\sigma_{\min,k}$ is given by

$$\boldsymbol{\lambda} = -\boldsymbol{Q}_k\boldsymbol{P}_k^{+}\boldsymbol{w}_k \tag{32}$$

where $\boldsymbol{P}_k^{+}$ is the pseudoinverse of $\boldsymbol{P}_k$ and $\boldsymbol{Q}_k$ is a matrix whose entries are 0's or 1's [18].

From the above, we reason a new result as below.

Proposition 5. *If the entries of* $\boldsymbol{P}_k^{+}\boldsymbol{w}_{k+1}$ *are not positive, then* $\boldsymbol{P}_{k+1} = \boldsymbol{P}_k$ *and* $\boldsymbol{Q}_{k+1} = \boldsymbol{Q}_k$.

Proof. The constraints of the problem (31) define a family of closed half-spaces. The intersection of them gives the polyhedral set $\mathscr{W}^*$ dual to the set $\mathscr{W}$. The objective function of (31) is a linear functional and attains its maximum value at an extreme point, namely $\boldsymbol{z}_k$, of $\mathscr{W}^*$ if it is bounded on $\mathscr{W}^*$. Then $-\boldsymbol{w}_k$ can be written as a nonnegative combination of the elements of $\mathscr{W}$ dual to $\boldsymbol{z}_k$, namely the columns of $\boldsymbol{P}_k$. If $\boldsymbol{P}_k^{+}\boldsymbol{w}_{k+1}$ is not positive, then $-\boldsymbol{w}_{k+1}$ can be expressed by a nonnegative combination of the columns of $\boldsymbol{P}_k$, and the function $-\boldsymbol{w}_{k+1}^{\mathrm{T}}\boldsymbol{z}$ on $\mathscr{W}^*$ reaches the maximal at the point dual to the columns of $\boldsymbol{P}_k$, which is just $\boldsymbol{z}_k$. Hence $\boldsymbol{P}_{k+1} = \boldsymbol{P}_k$ and $\boldsymbol{Q}_{k+1} = \boldsymbol{Q}_k$. □

Proposition 5 allows us to calculate $\sigma_{\min,k+1}$ without solving (31). Especially in computing $Q(G, \boldsymbol{w}_{\mathrm{ext}})$, $\boldsymbol{w}_{\mathrm{ext}}$ at t_k and t_{k+1} differs slightly, and probably $\boldsymbol{P}_{k+1} = \boldsymbol{P}_k$ and $\boldsymbol{Q}_{k+1} = \boldsymbol{Q}_k$. As a result, $\sigma_{\min,k+1}$ can be directly computed by (28) and (32).

Algorithm 2 (*Computing the quality criterion* $Q(G)$)

Step 1. Let $k = 1$.
Step 2. Compute $\sigma_{\min,k}$ and $\boldsymbol{z}_k$ by solving (31) w.r.t. $\boldsymbol{w}_k$. If the problem (31) does not have a feasible solution, then put out $Q(G) = -1$ and terminate the algorithm, which implies that $\boldsymbol{w}_k$ cannot be equilibrated by the grasp.
Step 3. Compute $\boldsymbol{P}_k^{+}$ and $\boldsymbol{Q}_k$.
Step 4. If $k + 1 > K$, then put out $Q(G) = \max_{k=1,2,\ldots,K}\sigma_{\min,k}$ and the algorithm ends successfully.
Step 5. If $\boldsymbol{P}_k^{+}\boldsymbol{w}_{k+1}$ is not positive, then let $\boldsymbol{P}_{k+1} = \boldsymbol{P}_k$ and $\boldsymbol{Q}_{k+1} = \boldsymbol{Q}_k$, and compute $\sigma_{\min,k+1}$ by (28) and (32); set $k = k + 1$ and return to Step 4. Otherwise, set $k = k + 1$ and return to Step 2.

The above formulation and computation of grasp quality criterion can be readily applied to frictionless point contact, since mathematically this kind of contact is a special case of frictional point contact [3]. By linearizing the soft finger contact constraint [38], they can also be extended to such contact.

4.3. Algorithm for computing the globally optimal grasp

After Algorithm 1 turns out the feasible grasps, finding the global optimum among them needs just to compare the values of $Q(G)$ for them and pick out the one for which $Q(G)$ is the smallest. To avoid exhaustively computing $Q(G)$ by Algorithm 2, a pretty trick is used (see Step 4), which speeds up the comparison.

Algorithm 3 (*Seeking the globally optimal grasp*)

Step 1. Use Algorithm 1 to find all feasible grasps, acting as candidates (*note*: geometrically equivalent grasps cannot be omitted because their performance qualities w.r.t. a task are probably different).

Step 2. Compute $Q(G)$ by Algorithm 2 for a candidate, marked by $\widehat{G}$. Let $\hat{k}$ be the point, for which $Q(G) = \sigma_{\min,\hat{k}}$. Let $\hat{Q} = Q(\widehat{G})$.
Step 3. If $\hat{Q} < 0$, then remove $\widehat{G}$ from the candidates and return to Step 2.
Step 4. Compute $\sigma_{\min,\hat{k}}$ by solving (31) w.r.t. $\boldsymbol{w}_{\hat{k}}$ for the other candidates (*note*: $\boldsymbol{w}_j, j = 1, 2, \ldots, n$ are calculated for each candidate respectively). If (31) for a candidate does not have solution, then the candidate cannot equilibrate $\boldsymbol{w}_{\hat{k}}$. If $\sigma_{\min,\hat{k}}$ for a candidate is not less than $\hat{Q}$, neither is $Q(G)$. Thus such candidates can be removed without computing their $Q(G)$.
Step 5. Search for a candidate whose $Q(G)$ is less than $\hat{Q}$ in the remaining. Remove those encountered in the searching process, for which $Q(G) < 0$ or $Q(G) \geqslant \hat{Q}$. If no such remainders exist, then $\widehat{G}$ is the globally optimal grasp; otherwise, update $\hat{k}, \hat{Q}$, and $\widehat{G}$, and return to Step 4.

5. Case study: a robot hand with three fingers and one palm

Fig. 3 sketches a hand, whose palm is fixed at the end of an axle and fingers are equally hinged around the axle and driven by a single actuator not shown. It looks somewhat humanoid, but all the balls are fixed joints except that the three balls attached to the axle are hinges. Thus the hand has only 1 DOF. Contacts with an object are made by only the fingertips, namely the hemispheres at the ends of fingers. Each one is of radius $r = 5$ mm. Let $\boldsymbol{p}_{iH}$, $i = 1, 2, 3$ be the position vectors of the ends of fingers relative to frame $\mathscr{F}_H$, which are expressed by

$$\begin{aligned}
\boldsymbol{p}_{1H}(\beta_1) &= [\,L\sin\beta_1 + R \quad 0 \quad L\cos\beta_1\,]^{\mathrm{T}}\\
\boldsymbol{p}_{2H}(\beta_2) &= [\,(L\sin\beta_2 + R)\cos(2\pi/3) \quad (L\sin\beta_2 + R)\sin(2\pi/3) \quad L\cos\beta_2\,]^{\mathrm{T}}\\
\boldsymbol{p}_{3H}(\beta_3) &= [\,(L\sin\beta_3 + R)\cos(4\pi/3) \quad (L\sin\beta_3 + R)\sin(4\pi/3) \quad L\cos\beta_3\,]^{\mathrm{T}}
\end{aligned}$$

where $L = 60$ mm, $R = 10$ mm, and $\beta_1, \beta_2, \beta_3$ are the rotation angles of the fingers, as indicated in Fig. 3. Let $\boldsymbol{d}_{iH}$, $i = 1, 2, 3$ be the vectors giving the directions of fingertips, which relative to frame $\mathscr{F}_H$ are

$$\begin{aligned}
\boldsymbol{d}_{1H}(\beta_1) &= [\,\sin(\beta_1 - \alpha) \quad 0 \quad \cos(\beta_1 - \alpha)\,]^{\mathrm{T}}\\
\boldsymbol{d}_{2H}(\beta_2) &= [\,\sin(\beta_2 - \alpha)\cos(2\pi/3) \quad \sin(\beta_2 - \alpha)\sin(2\pi/3) \quad \cos(\beta_2 - \alpha)\,]^{\mathrm{T}}\\
\boldsymbol{d}_{3H}(\beta_3) &= [\,\sin(\beta_3 - \alpha)\cos(2\pi/3) \quad \sin(\beta_3 - \alpha)\sin(2\pi/3) \quad \cos(\beta_3 - \alpha)\,]^{\mathrm{T}}
\end{aligned}$$

where $\alpha = \pi/3$.

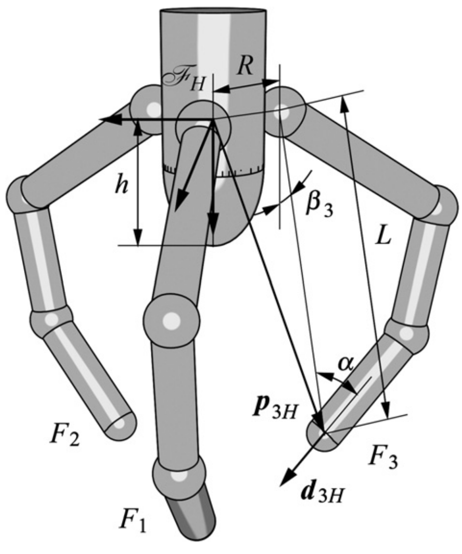

Fig. 3. Robot hand as a case study.

To grasp an object, first, let the palm contact the object such that the z-axis of frame $\mathscr{F}_H$ is parallel to the inward normal at contact. Denote the contact position by $\boldsymbol{r}_p$. Then $\boldsymbol{p}_{OH}$ and $\boldsymbol{R}_{OH}$ can be adopted as

$$\begin{aligned}
&\boldsymbol{p}_{OH}(\phi,\psi) = \boldsymbol{r}_p(\phi,\psi) - h\boldsymbol{n}_p(\phi,\psi)\\
&\boldsymbol{R}_{OH}(\phi,\psi,\theta) = [\boldsymbol{o}_p \quad \boldsymbol{t}_p \quad \boldsymbol{n}_p]\boldsymbol{R}(\theta)
\end{aligned}$$

where ϕ and ψ are the parameters of the object surface, $h = 30$ mm, and $\boldsymbol{R}(\theta) \in \mathbb{R}^{3\times 3}$ gives a rotation of the hand about $\boldsymbol{n}_p$:

$$\boldsymbol{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\theta \in [0, 2\pi/3)$. Next, actuate the fingers to approach the object. For easier computation, we formulate the fingertip F_i in frame $\mathscr{F}_H$ as a sphere of radius r centered at $\boldsymbol{p}_{iH}$, instead of a hemisphere. Then

$$p_{F_{iO}}(-\boldsymbol{z}) = -\boldsymbol{z}^{\mathrm{T}}(\boldsymbol{p}_{OH} + \boldsymbol{R}_{OH}\boldsymbol{p}_{iH}) + r$$

From Propositions 3 and 4, a feasible grasp exists w.r.t. ϕ, ψ, and θ if and only if the following conditions are satisfied:

1. There exist $\hat{\beta}_i$ and $\hat{\boldsymbol{z}}_i$ for all $i = 1,2,3$ such that $\rho(F_{iO}, O) = p_O(\hat{\boldsymbol{z}}_i) + p_{F_{iO}}(-\hat{\boldsymbol{z}}_i) = 0$.
2. $\hat{\boldsymbol{z}}_i^{\mathrm{T}}\boldsymbol{R}_{OH}\boldsymbol{d}_{iH} < 0$ for all $i = 1,2,3$.
3. $g(\hat{\beta}_1,\hat{\beta}_2,\hat{\beta}_3) = (\hat{\beta}_1 - \hat{\beta}_2)^2 + (\hat{\beta}_2 - \hat{\beta}_3)^2 + (\hat{\beta}_3 - \hat{\beta}_1)^2 = 0$.

Condition 1 can be fulfilled by means of the bisection method with (16). It together with condition 2 ensures that the fingertip F_i contacts the object by its open hemisphere. Since the fingers share one actuator, condition 3 finally makes certain the feasibility of a grasp. Furthermore, if the above conditions are satisfied, then the feasible grasp w.r.t. ϕ, ψ, and θ can be written as

$$G = \left\{\boldsymbol{r}_i = \boldsymbol{p}_{OH} + \boldsymbol{R}_{OH}\boldsymbol{p}_{iH} - r\hat{\boldsymbol{z}}_i, \boldsymbol{r}_4 = \boldsymbol{r}_p | i = 1,2,3\right\}$$

The vector $-\hat{\boldsymbol{z}}_i$ is just the unit normal at the contact point $\boldsymbol{r}_i$ towards the interior of the object O.

From the above arguments, the feasible grasps on an object can be sought in the domains of ϕ, ψ, θ. The proposed algorithms are implemented using MATLAB on a PC with P4 2.8 GHz processor, 1 MB cache memory, and 512 MB RAM. Assume that the friction coefficient $\mu_i = 0.2$ and the force upper bound $f_i^{\mathrm{U}} = 10$ N for each contact. Each friction cone is linearized into a 10-side polyhedral cone, i.e., $l = 10$ in (26).

Example 1. It is required to manipulate a bulb (Fig. 4), whose surface contains a sphere S of radius $R_0 = 20$ mm and a cone C. The origin of frame $\mathscr{F}_O$ is selected at the center of S. The cone C is expressed in frame $\mathscr{F}_O$ by

$$C = \operatorname{conv}\left(\bigcup_{k=1}^{2}\left\{\boldsymbol{r} \in \mathbb{R}^3 \middle| \sqrt{r_x^2 + r_y^2} = R_k, r_z = h_k\right\}\right)$$

where $R_1 = R_0\cos\alpha_0, h_1 = R_0\sin\alpha_0, R_2 = R_1 + H_0\tan\alpha_0, h_2 = h_1 - H_0, \alpha_0 = -\pi/6$, and $H_0 = 18$ mm. Two dynamic external wrenches will be applied on the bulb, which are specified in frame F_O by

$$\boldsymbol{w}_{\mathrm{ext}}^a = \begin{bmatrix} 0.5\cos 2.4\pi t + 10 \\ (2 + 0.5\sin 2.4\pi t)\sin 0.4\pi t \\ (2 + 0.5\sin 2.4\pi t)\cos 0.4\pi t \\ \cos(\sin 1.2\pi t)\sin(\cos 0.4\pi t) \\ \sin(\sin 1.2\pi t) \\ \cos(\sin 1.2\pi t)\cos(\cos 0.4\pi t) \end{bmatrix} \quad \text{and} \quad \boldsymbol{w}_{\mathrm{ext}}^b = \begin{bmatrix} 0.5\cos 2.4\pi t - 10 \\ (2 + 0.5\sin 2.4\pi t)\sin 0.4\pi t \\ (2 + 0.5\sin 2.4\pi t)\cos 0.4\pi t \\ \cos(\sin 1.2\pi t)\sin(\cos 0.4\pi t) \\ \sin(\sin 1.2\pi t) \\ \cos(\sin 1.2\pi t)\cos(\cos 0.4\pi t) \end{bmatrix}$$

They are periodical and their periods are both 5 s. We take 500 sampling times, i.e., $K = 500$ in (23).

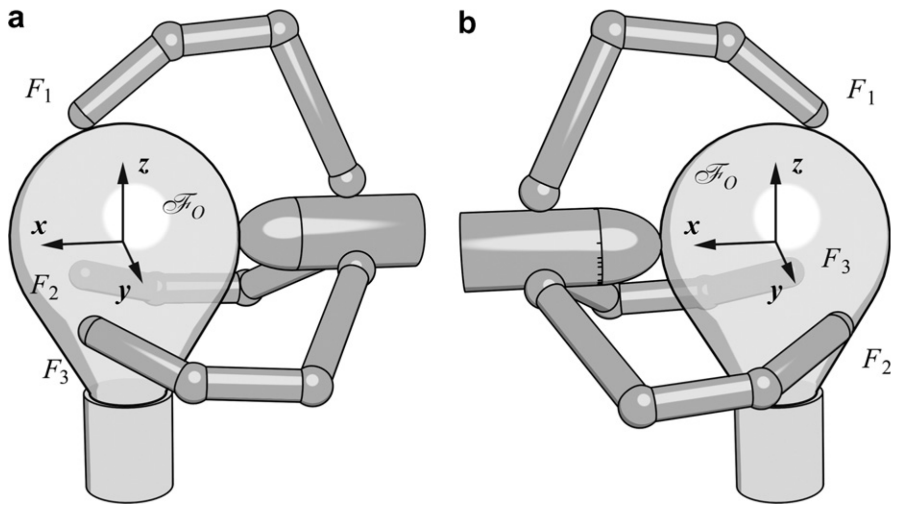

Fig. 4. Optimal grasps on a bulb for different manipuation task.

Using Theorem 4 and setting $z = [\cos\alpha_1\cos\alpha_2 \quad \cos\alpha_1\sin\alpha_2 \quad \sin\alpha_1]^{\mathrm{T}}$, we obtain

$$p_S(z) = R_0$$

$$p_C(z) = \begin{cases} R_2\cos\alpha_1 + h_2\sin\alpha_1, & \text{if } \alpha_1 < \alpha_0 \\ R_0, & \text{if } \alpha_1 = \alpha_0 \\ R_1\cos\alpha_1 + h_1\sin\alpha_1, & \text{if } \alpha_1 > \alpha_0 \end{cases}$$

and then

$$p_O(z) = \max\{p_S(z), p_C(z)\} = \begin{cases} R_2\cos\alpha_1 + h_2\sin\alpha_1, & \text{if } \alpha_1 < \alpha_0 \\ R_0, & \text{if } \alpha_1 \geqslant \alpha_0 \end{cases}$$

Accordingly, the L_2 distance between the finger F_{iO} and the bulb can be computed by (16).

Locate the palm on S. Then

$$\boldsymbol{r}_p = [R_0\cos\phi\cos\psi \quad R_0\cos\phi\sin\psi \quad R_0\sin\phi]^{\mathrm{T}}, \quad \boldsymbol{n}_p = -[\cos\phi\cos\psi \quad \cos\phi\sin\psi \quad \sin\phi]^{\mathrm{T}}$$
$$\boldsymbol{o}_p = [-\sin\phi\cos\psi \quad -\sin\phi\sin\psi \quad \cos\phi]^{\mathrm{T}}, \quad \boldsymbol{t}_p = [-\sin\psi \quad \cos\psi \quad 0]^{\mathrm{T}}$$

where $\phi \in [0, \pi/2]$ and $\psi \in [0, 2\pi)$. The steps of ϕ, ψ, and θ are taken to be $\pi/8$, $\pi/4$, and $\pi/12$, respectively. Using Algorithm 1 to search the domains for ϕ, ψ, and θ satisfying the above three conditions, we obtain 176 feasible grasps on the bulb with the CPU time of 55.40 min.

Running Algorithm 3 w.r.t. $\boldsymbol{w}_{\mathrm{ext}}^a$ yields the optimal grasp $\widehat{G}^a$ (Fig. 4a) at $\phi = 0, \psi = \pi$, and $\theta = 0$, for which $Q(\hat{G}^a, \boldsymbol{w}_{\mathrm{ext}}^a) = 0.8322$. Then $\hat{\beta}_1 = \hat{\beta}_2 = \hat{\beta}_3 = 0.0724\pi$ and the contact positions $\boldsymbol{r}_1 = [6.764 \quad 0 \quad 18.820]^{\mathrm{T}}$, $\boldsymbol{r}_2 = [6.764 \quad -16.299 \quad -9.410]^{\mathrm{T}}$, $\boldsymbol{r}_3 = [6.764 \quad 16.299 \quad -9.410]^{\mathrm{T}}$, and $\boldsymbol{r}_4 = [-20 \quad 0 \quad 0]^{\mathrm{T}}$. Running Algorithm 3 w.r.t. $\boldsymbol{w}_{\mathrm{ext}}^b$, we obtain grasp $\widehat{G}^b$ (Fig. 4b) at $\phi = 0, \psi = 0$, and $\theta = 0$, for which $Q(\widehat{G}^b, \boldsymbol{w}_{\mathrm{ext}}^b) = 0.8321$. At that time, $\hat{\beta}_1 = \hat{\beta}_2 = \hat{\beta}_3 = 0.0724\pi$, and $\boldsymbol{r}_1 = [-6.764 \quad 0 \quad 18.820]^{\mathrm{T}}, \boldsymbol{r}_2 = [-6.764 \quad 16.299 \quad -9.410]^{\mathrm{T}}$, $\boldsymbol{r}_3 = [-6.764 \quad -16.299 \quad -9.410]^{\mathrm{T}}$, and $r_4 = [20 \quad 0 \quad 0]^{\mathrm{T}}$. The required CPU times are 113.08 min and 126.21 min.

Example 2. The object O to be manipulated is a bottle (Fig. 5), which consists of an intercepted ellipsoid E and the spherical extension H of a hexahedron. The origin of frame $\mathscr{F}_O$ is selected at the center of E. The piece of surface E can be formulated in frame $\mathscr{F}_O$ as

$$E = \mathrm{conv}\Big\{[a\cos\gamma_1\cos\gamma_2 \quad a\cos\gamma_1\sin\gamma_2 \quad b\sin\gamma_1]^{\mathrm{T}} | -\pi/6 \leqslant \gamma_1 \leqslant \pi/6, 0 \leqslant_2 \leqslant 2\pi\Big\}$$

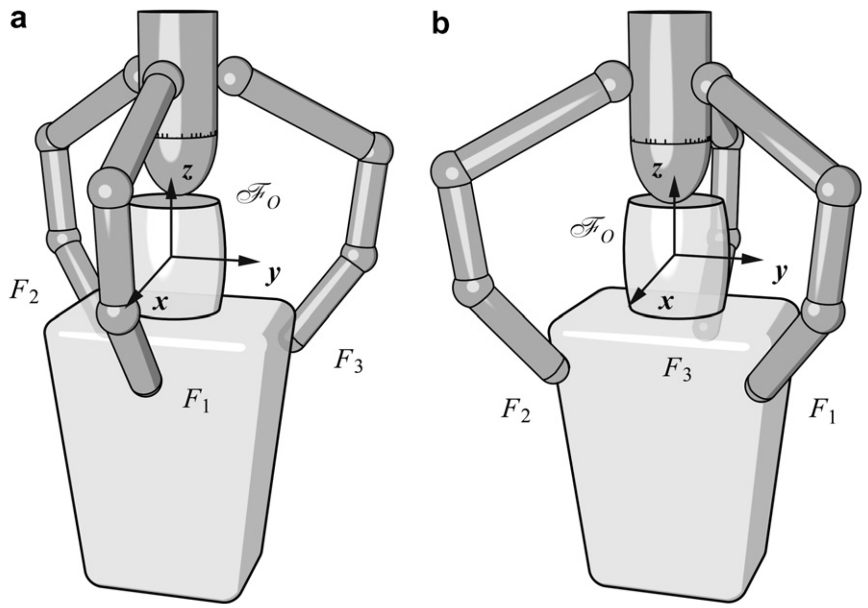

Fig. 5. Optimal grasps on a bottle for different manipulation tasks.

where $a = 10$ mm and $b = 20$ mm. The vertices of the hexahedron are

$$\boldsymbol{v}_1 = [12 \quad 16 \quad -12]^{\mathrm{T}}, \quad \boldsymbol{v}_2 = [-12 \quad 16 \quad -12]^{\mathrm{T}}, \quad \boldsymbol{v}_3 = [-12 \quad -16 \quad -12]^{\mathrm{T}}, \quad \boldsymbol{v}_4 = [12 \quad -16 \quad -12]^{\mathrm{T}}$$
$$\boldsymbol{v}_5 = [8 \quad 12 \quad -50]^{\mathrm{T}}, \quad \boldsymbol{v}_6 = [-8 \quad 12 \quad -50]^{\mathrm{T}}, \quad \boldsymbol{v}_7 = [-8 \quad -12 \quad -50]^{\mathrm{T}}, \quad \boldsymbol{v}_8 = [8 \quad -12 \quad -50]^{\mathrm{T}}$$

The radius of its spherical extension is $r_0 = 2$ mm. Let $\boldsymbol{z} = [\cos\alpha_1\cos\alpha_2 \quad \cos\alpha_1\sin\alpha_2 \quad \sin\alpha_1]^{\mathrm{T}}$. Then

$$p_E(\boldsymbol{z}) = \begin{cases} \sqrt{a^2\cos^2\alpha_1 + b^2\sin^2\alpha_1}, & \text{if } -\alpha_0 \leqslant \alpha_1 \leqslant \alpha_0 \\ a\cos(\pi/6)\cos\alpha_1 + b\sin(\pi/6)\sin|\alpha_1|, & \text{if } \alpha_1 < -\alpha_0 \text{ or } \alpha_1 > \alpha_0 \end{cases}$$
$$p_H(\boldsymbol{z}) = \max_{k=1,2,\ldots,8} \boldsymbol{z}^{\mathrm{T}}\boldsymbol{v}_k - r_0$$

where $\alpha_0 = \tan^{-1}(a\tan(\pi/6)/b)$. From (8), the L_2 distance between F_{iO} and O can be computed by

$$\rho(F_{iO}, O) = \min\{\rho(F_{iO}, E), \rho(F_{iO}, H)\}$$

where $\rho(F_{iO}, E)$ and $\rho(F_{iO}, H)$ are computed by (16) w.r.t. E and H, respectively.

Let the palm make contact with the upper face of E. Then

$$\boldsymbol{r}_p = [\phi\cos\psi \quad \phi\sin\psi \quad 10]^{\mathrm{T}}, \quad \boldsymbol{n}_p = [0 \quad 0 \quad -1]^{\mathrm{T}}, \quad \boldsymbol{o}_p = [0 \quad 1 \quad 0]^{\mathrm{T}}, \quad \boldsymbol{t}_p = [1 \quad 0 \quad 0]^{\mathrm{T}}$$

where $\phi \in [0, 8]$ and $\psi \in [0, 2\pi)$. The steps of ϕ, ψ, and θ are taken to be 2, $\pi/8$, and $\pi/12$, respectively. By Algorithm 1 we find 110 feasible grasps on the bottle with the CPU time of 184.65 min.

Suppose that the external wrench $\boldsymbol{w}_{\mathrm{ext}} = [f_x \quad f_y \quad f_z \quad m_x \quad m_y \quad m_z]^{\mathrm{T}}$ is limited by $f_x \in [-2, 1]$, $f_y \in [-1, 1], f_z \in [-3, 1], m_x \in [-1, 1], m_y \in [-1, 1]$, and $m_z \in [-1, 1]$. Thus the external wrench set $W^a_{\mathrm{ext}}$is given by the convex hull of 64 points in the wrench space. Using Algorithm 3, we find the optimal grasp $\widehat{G}^a$ (Fig. 5a) with $Q(\widehat{G}^a, W^a_{\mathrm{ext}}) = 0.9844$ at $\phi = 4, \psi = \pi, \theta = \pi/2$. Then $\hat{\beta}_1 = \hat{\beta}_2 = \hat{\beta}_3 = 0.0614\pi, \boldsymbol{r}_1 = [11.326 \quad 0 \quad -18.576]^{\mathrm{T}}, \boldsymbol{r}_2 = [-11.179 \quad -15.777 \quad -18.567]^{\mathrm{T}}, \boldsymbol{r}_3 = [-11.178 \quad 15.778 \quad -18.567]^{\mathrm{T}}$, and $\boldsymbol{r}_4 = [-4 \quad 0 \quad 10]^{\mathrm{T}}$. Changing the limitations of f_x into $f_x \in [-1, 2]$, we have another set W^b_{ext}. Running Algorithm 3 again yields the optimal grasp $\widehat{G}^b$ (Fig. 5b) with $Q(\widehat{G}^b, W^b_{\mathrm{ext}}) = 0.9844$ at $\phi = 4, \psi = 0, \theta = \pi/6$. Then $\hat{\beta}_1 = \hat{\beta}_2 = \hat{\beta}_3 = 0.0614\pi$, $\boldsymbol{r}_1 = [11.177 \quad 15.779 \quad -18.567]^{\mathrm{T}}, \boldsymbol{r}_2 = [11.177 \quad -15.779 \quad -18.567]^{\mathrm{T}}$, $\boldsymbol{r}_3 = [-11.326 \quad 0 \quad -18.576]^{\mathrm{T}}$, and $\boldsymbol{r}_4 = [4 \quad 0 \quad 10]^{\mathrm{T}}$. The CPU times for yielding $\widehat{G}^a$ and $\widehat{G}^b$ are 118.70 min and 128.64 min, respectively.

In Example 1, $\widehat{G}^a$ and $\widehat{G}^b$ are geometrically equivalent, but by Algorithm 2 we see $Q(\widehat{G}^a, \boldsymbol{w}^b_{\text{ext}}) = 1.1020 > Q(\widehat{G}^b, \boldsymbol{w}^b_{\text{ext}}) = 0.8321$ and $Q(\widehat{G}^b, \boldsymbol{w}^a_{\text{ext}}) = 1.1022 > Q(\widehat{G}^a, \boldsymbol{w}^a_{\text{ext}}) = 0.8322$. Both $Q(\widehat{G}^a, \boldsymbol{w}^b_{\text{ext}})$ and $Q(\widehat{G}^b, \boldsymbol{w}^a_{\text{ext}})$ are over unity. A similar result is obtained in Example 2, where $Q(\widehat{G}^a, W^b_{\text{ext}}) = 1.3321 > Q(\widehat{G}^b, W^b_{\text{ext}}) = 0.9844$ and $Q(\widehat{G}^b, \boldsymbol{w}^a_{\text{ext}}) = 1.3321 > Q(\widehat{G}^a, W^a_{\text{ext}}) = 0.9844$. This means that an optimal grasp for wrench a is a bad grasp for wrench b, and vice versa.

6. Conclusion and future work

This paper suggests remedies for some significant weaknesses in the previous OGP research. First, using the L_2 distance between sets, we present a method of seeking feasible grasps on an object conforming to a hand. This method can compute various forms of grasps (e.g. fingertip grasp and power grasp) on a family of objects (comprising some convex parts and having round corners) conforming to different grasping mechanisms (especially multifingered hands). It should be noted, however, that the decomposition of a nonconvex object into convex parts still relies on human intervention, rather than artificial intelligence. Besides, a task-oriented grasp quality criterion together with the computational method is given. It is invariant under a change of unit and coordinate frame. Finally, we develop an algorithm for picking the globally optimal grasp among the feasible ones by the criterion. The resulting grasp possesses the best performance quality in a task.

Future work may focus on other practical conditions, such as the stiffness which plays an important role in the compliance of a grasp. Attention could also be concentrated on quality criteria. In addition to the efficiency of a grasp to equilibrate external wrenches in a task, the properties summarized in [32] deserve notice as well, such as dexterity and stability. Grasp planning with multiple optimization goals was rarely referred to until now. In addition, when none of the feasible grasps is competent for the whole task, one has to divide the task into several phases and use different grasps. Then the strategy of regrasping, i.e., changing the grasp from one phase to another, should be considered.

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