CHAPTER 2

Rigid Body Motion ,
Robot Kinematics of Velocity,
and Robot Statics

After this chapter, the students are expected to learn the following:

1. Relate time derivatives of position and orientation representations with translational and angular velocities.

2. Transform velocities in different spaces

3. Relate joint velocities with end-effector velocities

After this chapter, the students are expected to learn the following:

4. Understand the concept of Jacobians

5. Solve the forward and inverse kinematics of velocity

6. Understand robot singularities

After this chapter, the students are expected to learn the following:

7. Static force/torque transformations between frames

8. Static force/torque transformations between task space and joint space

9. Understand redundancy and how to deal with them

Translational Velocities

$^{A}\mathbf{u}_{B} \in \Re^{3x1}$ = translational velocity of frame B (i.e., origin of frame B) relative of frame A

$$^{A}\mathbf{u}_{B} = \frac{d}{dt}\,^{A}\mathbf{p}_{B} = \lim_{\Delta t \to 0} \frac{^{A}\mathbf{p}_{B}(t+\Delta t) - \,^{A}\mathbf{p}_{B}(t)}{\Delta t}$$

↳ "frame of differentiation" is A

Velocity, like any vector may be expressed in another frame, say W

$$^{W}_{A}\mathbf{u}_{B} = \,^{W}\mathbf{r}_{A}\,^{A}\mathbf{u}_{B}$$

Translational Velocities

In general, velocity vector depends on 2 frames:

- frames of differentiation – A – leading subscript – this is the frame where the velocity of B is instantaneously computed from (can be thought of as velocity reference point)
- frame resulting vector is expressed in – W – leading superscript

When leading subscript is omitted, it is implied that the velocity is relative to some understood universal frame of reference.

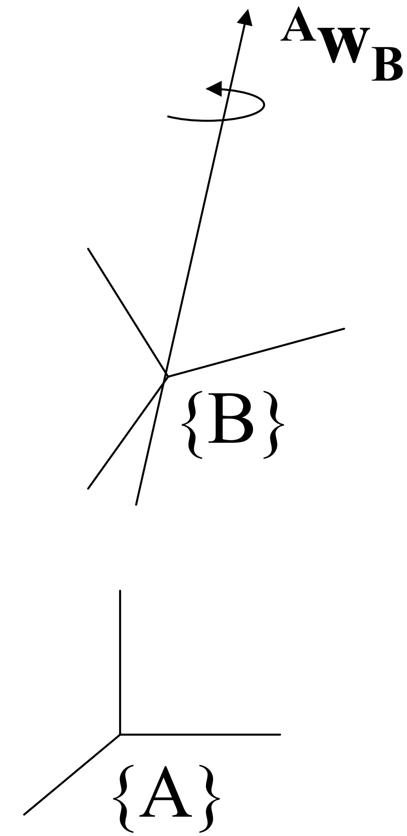
A missing leading superscript implies a generic frame of expression.

Rotational Velocities

- At a certain instant, frame B has an orientation $^{A}\mathbf{R}_{B}$ and its rotational motion may be represented by the rotational (angular) velocity vector

$^{A}\mathbf{w}_{0} \in \Re^{3x1}$

- unit vector along $^{A}\mathbf{w}_{B}$ = instantaneous axis of rotation $= {}^{A}\mathbf{k}_{B}$
- magnitude of $^{A}\mathbf{w}_{B}$ = speed of rotation



- $^{A}\mathbf{w}_{B}$ is related to $\frac{d}{dt}{}^{A}\mathbf{R}_{B} = {}^{A}\dot{\mathbf{R}}_{B} = \lim_{\Delta t \to 0} \frac{{}^{A}\mathbf{R}_{B}(t+\Delta t) - {}^{A}\mathbf{R}_{B}(t)}{\Delta t}$

Rotational Velocities

Let **Rot (k, d**θ **)** = incremental change in rotation.

$$ {}^{A}\mathbf{w}_{B} = {}^{A}\mathbf{k}_{B}\,\dot{\theta} = {}^{A}\mathbf{k}_{B}\,\frac{d\theta}{dt} $$

$$ \therefore\quad {}^{A}\mathbf{R}_{B} + \Delta\,{}^{A}\mathbf{R}_{B} = \underbrace{\mathbf{Rot}\,(\,\mathbf{k}\,,\,\mathbf{d}\theta\,)}\,{}^{A}\mathbf{R}_{B} $$

pre-multiplication since $d\theta\ {}^{A}\vec{\mathbf{k}}_{B}$ is expressed in the frame A (base)

Rot (k, θ **)** =

$$ \begin{pmatrix} k_x k_x \operatorname{vers}\theta + \cos\theta & k_y k_x \operatorname{vers}\theta - k_z \sin\theta & k_z k_x \operatorname{vers}\theta + k_y \sin\theta \\ k_x k_y \operatorname{vers}\theta + k_z \sin\theta & k_y k_y \operatorname{vers}\theta + \cos\theta & k_z k_y \operatorname{vers}\theta - k_x \sin\theta \\ k_x k_z \operatorname{vers}\theta - k_y \sin\theta & k_y k_z \operatorname{vers}\theta + k_x \sin\theta & k_z k_z \operatorname{vers}\theta + \cos\theta \end{pmatrix} $$

where versθ = (1 - cosθ)

Rotational Velocities

For a differential change (small) <u>dθ</u>

$$\cos(\theta + d\theta) = \cos\theta + \frac{\delta\cos\theta}{\delta\theta}d\theta = \cos\theta + (-\sin\theta)d\theta$$

for $\theta = 0$, $d\theta$ small

$$\cos(d\theta) \stackrel{\rightarrow}{=} 1 \quad \text{(approx)}$$

$$\sin(\theta + d\theta) = \sin\theta + \frac{\delta\sin\theta}{\delta\theta}d\theta = \sin\theta + \cos\theta(d\theta)$$

Rotational Velocities

for $\theta = 0$, $d\theta$ = small

$$\sin(d\theta) = 0 + 1d\theta$$

$$\overset{\rightarrow}{=} d\theta \quad \text{(approx)}$$

for $\theta = 0$, $d\theta$ = small

$$\text{vers}(d\theta) = 1 - \cos(d\theta) \overset{\rightarrow}{=} 0 \quad \text{(approx)}$$

Rotational Velocities

$$\therefore \textbf{Rot}\ (\mathbf{k}, \mathbf{d\theta}) = \begin{bmatrix} 1 & -k_z d\theta & k_y d\theta \\ k_z d\theta & 1 & -k_x d\theta \\ -k_y d\theta & k_x d\theta & 1 \end{bmatrix}$$

Back to:

$$\Delta {}^A R_B = \text{Rot}\,(k, d\theta)\ {}^A R_B - {}^A R_B$$
$$= \left[\text{Rot}\,(k, d\theta) - I\right] {}^A R_B$$

$$\Delta {}^A R_B = \begin{bmatrix} \phi & -k_z d\theta & k_y d\theta \\ k_z d\theta & \phi & -k_x d\theta \\ -k_y d\theta & k_x d\theta & \phi \end{bmatrix} {}^A R_B$$

Rotational Velocities

dividing by dt:

$$\frac{d^A\mathbf{R}_B}{dt} = \begin{bmatrix} \phi & -k_z \frac{d\theta}{dt} & k_y \frac{d\theta}{dt} \\ k_z \frac{d\theta}{dt} & \phi & -k_x \frac{d\theta}{dt} \\ -k_y \frac{d\theta}{dt} & k_x \frac{d\theta}{dt} & \phi \end{bmatrix} {}^A\mathbf{R}_B$$

But $${}^A\mathbf{w}_B = {}^A\mathbf{k}_B\,\dot{\theta} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \dot{\theta} = \begin{bmatrix} {}^A w_{Bx} \\ {}^A w_{By} \\ {}^A w_{Bz} \end{bmatrix}$$

Rotational Velocities

$$\therefore\ {}^{A}\dot{\mathbf{R}}_{\mathbf{B}} = \begin{bmatrix} 0 & -{}^{A}w_{Bz} & {}^{A}w_{By} \\ {}^{A}w_{Bz} & 0 & -{}^{A}w_{Bx} \\ -{}^{A}w_{By} & {}^{A}w_{Bx} & 0 \end{bmatrix} {}^{A}\mathbf{R}_{\mathbf{B}}$$

Let this be $\lfloor {}^{\mathbf{A}}\mathbf{w}_{\mathbf{B}} \mathrm{x} \rfloor =$
angular velocity tensor of ${}^{\mathbf{A}}\mathbf{w}_{\mathbf{B}}$

$${}^{\mathbf{A}}\dot{\mathbf{R}}_{\mathbf{B}} = \lfloor {}^{\mathbf{A}}\mathbf{w}_{\mathbf{B}} \mathrm{x} \rfloor\, {}^{\mathbf{A}}\mathbf{R}_{\mathbf{B}}$$

Rotational Velocities

As with any vector, the rotational velocity vector ${}^{A}\mathbf{w}_{B}$ may be expressed in another frame C:

$$\underbrace{{}^{C}_{A}\mathbf{w}_{B}} = {}^{C}\mathbf{R}_{A}\ {}^{A}\mathbf{w}_{B}$$

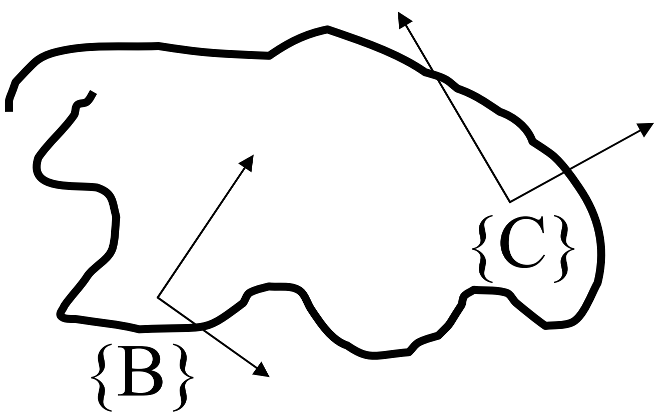
leading subscript A: frame the body is rotating relative to (frame & differentiation)

leading superscript C: frame of Expression

Rotational Velocities

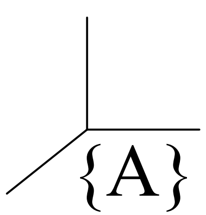
The equivalent matrix product representation is:

$$\lfloor {}^{C}_{A}\mathbf{w}_{B}\, x \rfloor = {}^{C}\mathbf{R}_{A} \lfloor {}^{A}\mathbf{w}_{B}\, x \rfloor\, {}^{A}\mathbf{R}_{C}$$



{B} & {C} are attached to the same rigid body which is rotating

$${}^{A}\mathbf{R}_{C} = {}^{A}\mathbf{R}_{B}\, {}^{B}\mathbf{R}_{C}$$



Diff with time $\longrightarrow$ $${}^{A}\dot{\mathbf{R}}_{C} = {}^{A}\mathbf{R}_{B}\, \cancel{{}^{A}\dot{\mathbf{R}}_{C}}^{\,\phi} + {}^{A}\dot{\mathbf{R}}_{B}\, {}^{B}\mathbf{R}_{C}$$

Rotational Velocities

$$\lfloor {}^{A}\mathbf{w}_{C} \times \rfloor \, {}^{A}\mathbf{R}_{C} = \lfloor {}^{A}\mathbf{w}_{B} \times \rfloor \, {}^{A}\mathbf{R}_{B} \, {}^{B}\mathbf{R}_{C}$$

$$\lfloor {}^{A}\mathbf{w}_{C} \times \rfloor \, \cancel{{}^{A}\mathbf{R}_{C}} = \lfloor {}^{A}\mathbf{w}_{B} \times \rfloor \, \cancel{{}^{A}\mathbf{R}_{C}}$$

$${}^{A}\mathbf{w}_{C} = {}^{A}\mathbf{w}_{B}$$

$\Rightarrow$ rotational velocity of rigid body is equal to rot. velocity of any frame attached to the rigid body.

The Orthonormal (Rotation) Matrix & Skew Symmetric Matrices

$\mathbf{R}\mathbf{R}^\mathbf{T} = \mathbf{I}$

$\mathbf{R}\dot{\mathbf{R}}^\mathbf{T} + \dot{\mathbf{R}}\mathbf{R}^\mathbf{T} = \underline{0}$

$(\dot{\mathbf{R}}\mathbf{R}^\mathbf{T})^\mathbf{T} + \dot{\mathbf{R}}\mathbf{R}^\mathbf{T} = \underline{0}$

Define $\mathbf{S} = \dot{\mathbf{R}}\mathbf{R}^\mathbf{T} = \lfloor \mathbf{w}\mathrm{x} \rfloor$

Then $\mathbf{S} + \mathbf{S}^\mathbf{T} = \underline{0}$

$\mathbf{S}$ = a skew symmetric matrix

The Orthonormal (Rotation) Matrix & Skew Symmetric Matrices

Skew symmetric matrix as vector cross product:

$$\text{Let } \mathbf{S} = \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix} \text{ and } \mathbf{w} = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$$

Then

$\mathbf{Sp} = \mathbf{w} \times \mathbf{p}$

where $\mathbf{p} \in \Re^{3x1}$ vector

Rotation Matrix

$$\dot{\mathbf{R}} = \lfloor \mathbf{w}\text{x} \rfloor \mathbf{R} \qquad \text{or}$$

↓

3x3 (time derivative of Rot. Matrix)

or

$$\begin{pmatrix} \dot{n} \\ \dot{o} \\ \dot{a} \end{pmatrix} = \begin{pmatrix} -\lfloor n\text{x} \rfloor \\ -\lfloor o\text{x} \rfloor \\ -\lfloor a\text{x} \rfloor \end{pmatrix} \mathbf{w}$$

$$\dot{\mathbf{x}}_{\mathbf{r}} = \mathbf{E}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}})\ \mathbf{w}$$

- $\dot{\mathbf{x}}_{\mathbf{r}}$: 9x1 representation of orientation
- $\mathbf{E}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}})$: 9x3 (a kind of Jacobian associated with representation)
- $\mathbf{w}$: 3x1 angular velocity

Rotation Matrix

- Trajectories are typically specified in terms of $\mathbf{x_r}$, and it is important to determine the angular velocities
- To solve for $\mathbf{w}$ given $\dot{\mathbf{x}}_\mathbf{r}$, need to solve 9 Equations with 3 unknowns $\longrightarrow$ overdetermined system
- Solution that minimizes $\| \mathbf{E_r}\,\mathbf{w} - \dot{\mathbf{x}}_\mathbf{r} \|$ is the left pseudo inverse, $\mathbf{E_r}^+$

$$\mathbf{w} = \mathbf{E_r}^+ \dot{\mathbf{x}}_\mathbf{r}$$

$$\mathbf{E_r}^+ = \underbrace{(\mathbf{E_r}^T \mathbf{E_r})^{-1}}_{\text{always exists}} \mathbf{E_r}^T$$

Rotation Matrix

But from definition of $\mathbf{E_r}$

$$\mathbf{E_r}^{\mathbf{T}}\,\mathbf{E_r} = \left(\lfloor -nx \rfloor^T \quad \lfloor -ox \rfloor^T \quad \lfloor -ax \rfloor^T \right) \begin{pmatrix} \lfloor -nx \rfloor \\ \lfloor -ox \rfloor \\ \lfloor -ax \rfloor \end{pmatrix}$$

$$= +\lfloor nx \rfloor^T \lfloor nx \rfloor + \lfloor ox \rfloor^T \lfloor ox \rfloor + \lfloor ax \rfloor^T \lfloor ax \rfloor$$

$$= 2\mathbf{I}_3$$

$\therefore$

$$\mathbf{E_r}^{+} = (\mathbf{E_r}^{\mathbf{T}}\,\mathbf{E_r})^{-1}\,\mathbf{E_r}^{\mathbf{T}}$$

$$= (2\mathbf{I}_3)^{-1}\,\mathbf{E_r}^{\mathbf{T}} = \tfrac{1}{2}\,\mathbf{E_r}^{\mathbf{T}}$$

//

very simple

Rotation Matrix

$$\mathbf{w} = \tfrac{1}{2}\, \mathbf{E_r}^{\mathbf{T}} \begin{pmatrix} \dot{n} \\ \dot{o} \\ \dot{a} \end{pmatrix} = \tfrac{1}{2}\,(\lfloor nx \rfloor^T \dot{n} + \lfloor ox \rfloor^T \dot{o} + \lfloor ax \rfloor^T \dot{a}$$

Exactly the same

Actually,

$$\dot{\mathbf{R}} = \lfloor \mathbf{w}\mathrm{x} \rfloor \mathbf{R}$$

$$\lfloor \mathbf{w}\mathrm{x} \rfloor = \dot{\mathbf{R}}\, \mathbf{R}^{\mathrm{T}}$$

Note: Free of Math. Singularities

$$\left.\begin{matrix} \dot{\mathbf{x}}_{\mathbf{r}} \longrightarrow \mathbf{w} \\ \mathbf{w} \longrightarrow \dot{\mathbf{x}}_{\mathbf{r}} \end{matrix}\right\} \text{always possible}$$

Euler Angle Rates & Angular Velocities

$$^{A}R_{B} = Rot\,(z, \alpha)\,Rot\,(y, \beta)\,Rot\,(z, \gamma)$$

$$\mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\dot{\alpha} + \mathbf{Rot}(z, \alpha)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\dot{\beta} + \mathbf{Rot}(z, \alpha)\,\mathbf{Rot}(y, \beta)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\dot{\gamma}$$

$$\mathbf{w} = \underbrace{\begin{pmatrix} 0 & -\sin\alpha & \cos\alpha\sin\beta \\ 0 & \cos\alpha & \sin\alpha\sin\beta \\ 1 & 0 & \cos\beta \end{pmatrix}}\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix}$$

$$\mathbf{w} = \mathbf{E_r}^{-1}\,\dot{\mathbf{x}}_{\mathbf{r}}$$

E_r → Jacobian transformation

Euler Angle Rates & Angular Velocities

or $\dot{\mathbf{x}}_\mathbf{r} = \mathbf{E}_\mathbf{r}\,\mathbf{w}$ note that $\mathbf{E}_\mathbf{r}$ does not always exist

$$\dot{\mathbf{x}}_\mathbf{r} = \underbrace{\begin{pmatrix} \dfrac{-\cos\alpha\cos\beta}{\sin\beta} & \dfrac{-\sin\alpha\cos\beta}{\sin\beta} & 1 \\ -\sin\alpha & \cos\alpha & 0 \\ \dfrac{\cos\alpha}{\sin\beta} & \dfrac{\sin\alpha}{\sin\beta} & 0 \end{pmatrix}}\,\mathbf{w}$$

If $\sin\beta = 0$, matrix does not exist

$\rightarrow$ Math. Singularity

$\mathbf{w} \rightarrow \dot{\mathbf{x}}_\mathbf{r}$ not always possible

Not all possible angular matrices can be represented

Problem with 3 parameter representations for orientation

Roll Pitch Yaw Rates & Angular Velocities

$${}^{A}\mathbf{R}_{B} = \mathbf{Rot}\ (\ \mathbf{z},\ \phi\ )\ \mathbf{Rot}\ (\ \mathbf{y},\ \theta\ )\ \mathbf{Rot}\ (\ \mathbf{x},\ \varphi)$$

$$w = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\dot{\phi} + \mathrm{Rot}(z,\phi)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\dot{\theta} + \mathrm{Rot}(z,\phi)\,\mathrm{Rot}(y,\theta)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\dot{\varphi}$$

$$w = \underbrace{\begin{pmatrix} 0 & -\sin\phi & \cos\phi\cos\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 1 & 0 & -\sin\theta \end{pmatrix}}_{\mathbf{E_r}^{-1}{}_{\mathbf{RPY}}}\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\varphi} \end{pmatrix}$$

$$\mathbf{w} = \mathbf{E_r}^{-1}\,\dot{\mathbf{x}}_{\mathbf{r}}$$

OR

$$\dot{\mathbf{x}}_{\mathbf{r}} = \mathbf{E_r}\,\mathbf{w}$$

Roll Pitch Yaw Ratio & Angular Velocities

$$\dot{x}_r = \begin{pmatrix} \frac{\cos\phi \sin\phi}{\cos\theta} & \frac{\sin^2\phi}{\cos\theta} & 1 \\ -\sin\phi & \cos\phi & 0 \\ \frac{\cos\phi}{\cos\theta} & \frac{\sin\phi}{\cos\theta} & 0 \end{pmatrix} w$$

If cosθ = 0, matrix does not exist

→ Math. Singularity

w → $\dot{\mathbf{x}}_{\mathbf{r}}$ not always possible

Not all possible angular matrix can be represented

This is a problem with 3 parameter representations for orientation

Quaternion Rates & Angular Velocities

$$\dot{\mathbf{R}}\,\mathbf{R}^T = \lfloor \mathbf{w}\mathrm{x} \rfloor$$

$\dot{\mathbf{R}} \rightarrow f(\dot\lambda_0, \dot\lambda_1, \dot\lambda_2, \dot\lambda_3)$

$\mathbf{R}^T \rightarrow f(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$

OR

$$\left.\begin{aligned} \lambda_0 &= \cos\theta/2 \\ \lambda_1 &= k_x \sin\theta/2 \\ \lambda_2 &= k_y \sin\theta/2 \\ \lambda_3 &= k_z \sin\theta/2 \end{aligned}\right\} \frac{d}{dt}\text{it}$$

$$\begin{pmatrix} \dot\lambda_0 \\ \dot\lambda_1 \\ \dot\lambda_2 \\ \dot\lambda_3 \end{pmatrix} = \frac{1}{2} \underbrace{\begin{pmatrix} -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_0 & \lambda_3 & -\lambda_2 \\ -\lambda_3 & \lambda_0 & \lambda_1 \\ \lambda_2 & -\lambda_1 & \lambda_0 \end{pmatrix}}_{} \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}$$

$$\dot{\mathbf{x}}_{\mathbf{r}} = \mathbf{E}_{\mathbf{r}}(\mathbf{x})\,\mathbf{w}$$

Quaternion

Quaternion Rates & Angular Velocities

always exist

$$\mathbf{w} = \mathbf{E_r(x)^+ \dot{x}_r}$$

$$\mathbf{E_r(x)^+ = [\ E_r(x)^T E_r(x)\ ]^{-1} E_r^T(x)} \quad \text{(left pseudo inverse)}$$

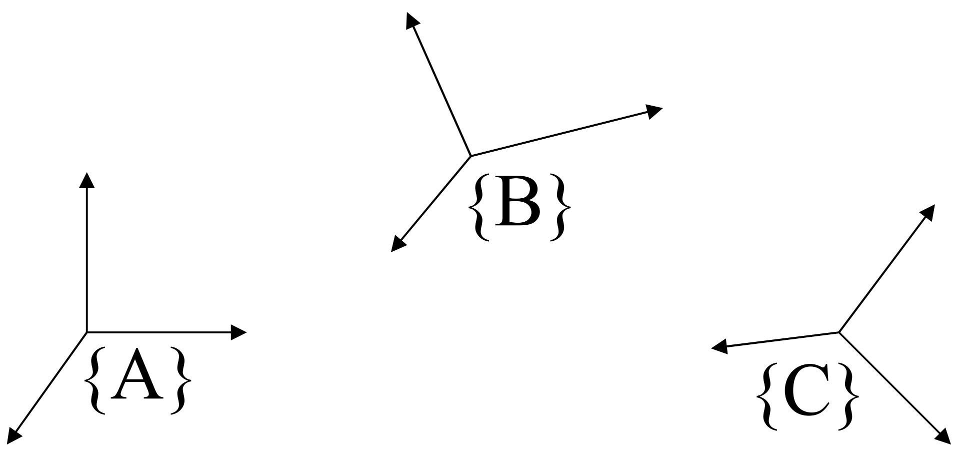
$$= 2\begin{pmatrix} -\lambda_1 & \lambda_0 & -\lambda_3 & \lambda_2 \\ -\lambda_2 & \lambda_3 & \lambda_0 & -\lambda_1 \\ -\lambda_3 & -\lambda_2 & \lambda_1 & \lambda_0 \end{pmatrix}$$

Note: Free of Math. Singularities

$$\left.\begin{matrix} \mathbf{w} \longrightarrow \mathbf{\dot{x}_r} \\ \mathbf{\dot{x}_r} \longrightarrow \mathbf{w} \end{matrix}\right\} \text{always possible}$$

Simultaneous Rotational & Translational Velocities

Given: Frames A, B & C



{B} & {C} in motion with respect to {A}

Find: Relationships between velocities

$$ {}^{A}\mathbf{p}_{C} = {}^{A}\mathbf{p}_{B} + {}^{A}\mathbf{R}_{B}\, {}^{B}\mathbf{p}_{C} $$

Simultaneous Rotational & Translational Velocities

Differentiating

$$ {}^{A}\mathbf{U}_{C} = {}^{A}\mathbf{U}_{B} + {}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{U}_{C} + \underbrace{{}^{A}\dot{\mathbf{R}}_{B}\,{}^{B}\mathbf{p}_{C}}$$

→ Contribution of rotational velocity of frame B to the translational velocity of C =

$$ {}^{A}\dot{\mathbf{R}}_{B}\,{}^{B}\mathbf{p}_{C} = \lfloor {}^{A}\mathbf{w}_{B} \times \rfloor\, {}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{p}_{C} = {}^{A}\mathbf{w}_{B} \times ({}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{p}_{C}) $$

$$ = -\,{}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{p}_{C} \times {}^{A}\mathbf{w}_{B} $$

$$ = {}^{A}\mathbf{w}_{B} \times ({}^{A}\mathbf{p}_{C} - {}^{A}\mathbf{p}_{B}) $$

$$ \therefore\ {}^{A}\mathbf{U}_{C} = {}^{A}\mathbf{U}_{B} + {}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{U}_{C} + {}^{A}\mathbf{w}_{B} \times ({}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{p}_{C}) $$

$$ = {}^{A}\mathbf{U}_{B} + {}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{U}_{C} + {}^{A}\mathbf{w}_{B} \times ({}^{A}\mathbf{p}_{C} - {}^{A}\mathbf{p}_{B}) $$

Simultaneous Rotational & Translational Velocities

$$ {}^{A}\mathbf{R}_{C} = {}^{A}\mathbf{R}_{B}\, {}^{B}\mathbf{R}_{C} $$

$$ {}^{A}\dot{\mathbf{R}}_{C} = {}^{A}\dot{\mathbf{R}}_{B}\, {}^{B}\mathbf{R}_{C} + {}^{A}\mathbf{R}_{B}\, {}^{B}\dot{\mathbf{R}}_{C} $$

$$ \lfloor {}^{A}\mathbf{w}_{C}\mathrm{x} \rfloor\, {}^{A}\mathbf{R}_{C} = \lfloor {}^{A}\mathbf{w}_{B}\mathrm{x} \rfloor\, {}^{A}\mathbf{R}_{B}\, {}^{B}\mathbf{R}_{C} + {}^{A}\mathbf{R}_{B} \lfloor {}^{B}\mathbf{w}_{C}\mathrm{x} \rfloor\, {}^{B}\mathbf{R}_{C} $$

$$ = \lfloor {}^{A}\mathbf{w}_{B}\mathrm{x} \rfloor\, {}^{A}\mathbf{R}_{C} + {}^{A}\mathbf{R}_{B} \lfloor {}^{B}\mathbf{w}_{C}\mathrm{x} \rfloor\, {}^{B}\mathbf{R}_{A}\, {}^{A}\mathbf{R}_{C} $$

$$ \lfloor {}^{A}\mathbf{w}_{C}\mathrm{x} \rfloor\, \cancel{{}^{A}\mathbf{R}_{C}} = \lfloor {}^{A}\mathbf{w}_{B}\mathrm{x} \rfloor\, \cancel{{}^{A}\mathbf{R}_{C}} + {}^{A}\mathbf{R}_{B} \lfloor {}^{B}\mathbf{w}_{C}\mathrm{x} \rfloor\, {}^{B}\mathbf{R}_{A}\, \cancel{{}^{A}\mathbf{R}_{C}} $$

$$ \lfloor {}^{A}\mathbf{w}_{C}\mathrm{x} \rfloor = \lfloor {}^{A}\mathbf{w}_{B}\mathrm{x} \rfloor + {}^{A}\mathbf{R}_{B} \lfloor {}^{B}\mathbf{w}_{C}\mathrm{x} \rfloor\, {}^{B}\mathbf{R}_{A} $$

Simultaneous Rotational & Translational Velocities

But

$$\lfloor {}^{A}_{B}\mathbf{w}_{C}\ \mathrm{x} \rfloor = {}^{A}\mathbf{R}_{B} \lfloor {}^{B}\mathbf{w}_{C}\mathrm{x} \rfloor\, {}^{B}\mathbf{R}_{A} \Leftrightarrow$$

$${}^{A}_{B}\mathbf{w}_{C} = {}^{A}\mathbf{R}_{B}\ {}^{B}\mathbf{w}_{C}$$ expressing vector in diff frame

$$\therefore\ \lfloor {}^{A}\mathbf{w}_{C}\mathrm{x} \rfloor = \lfloor {}^{A}\mathbf{w}_{B}\mathrm{x} \rfloor + \lfloor {}^{A}_{B}\mathbf{w}_{C}\ \mathrm{x} \rfloor$$

OR in vector form:

$${}^{A}\mathbf{w}_{C} = {}^{A}\mathbf{w}_{B} + {}^{A}\mathbf{R}_{B}\ {}^{B}\mathbf{w}_{C}$$

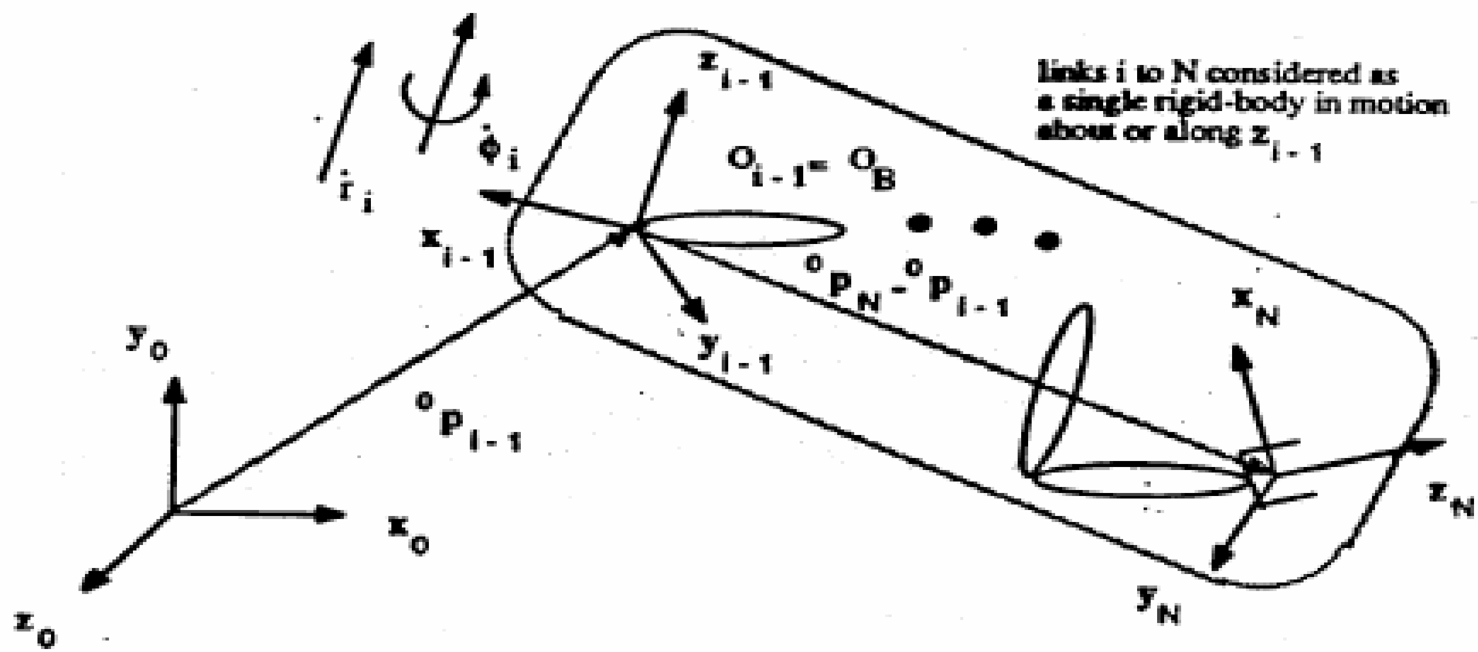
Note also (in homogeneous transformation)

$${}^{A}\dot{\mathbf{T}}_{B} = \left[\begin{array}{c|c} \lfloor {}^{A}\mathbf{w}_{B}\ \mathrm{x} \rfloor\, {}^{A}\mathbf{R}_{B} & {}^{A}\mathbf{U}_{B} \\ \hline 0 & 0 \end{array}\right]$$

Computation Of End-Effector Velocity

$$(6\text{x}1) \qquad \mathbf{v}_{\mathbf{N}} = \begin{pmatrix} u_N \\ w_N \end{pmatrix} = f\,(\,q, \dot{q}\,)$$

joint position (q), joint velocities ($\dot{q}$)



Computation Of End-Effector Velocity

Let us examine the contribution of the ith joint motion to end-effector velocity. We set all other joint velocities ϕ:

$\dot{\mathbf{q}}_C \neq 0$ $\quad\quad$ $\dot{\mathbf{q}}_1 = \dot{\mathbf{q}}_2 = \ldots = \dot{\mathbf{q}}_{i-1} = \dot{\mathbf{q}}_{i+1} = \ldots \dot{\mathbf{q}}_N = \phi$

so motion is occurring with respect to $\mathbf{z}_{i-1}$ axis

For joint i rotational

$$\mathbf{w}_i = \mathbf{z}_{i-1}\,\dot{\mathbf{q}}_i$$

$$\mathbf{u}_i = \mathbf{w}_i \times \mathbf{R}_{i-1}\,{}^{i-1}\mathbf{p}_N = \mathbf{z}_{i-1}\,\dot{\mathbf{q}}_i \times (\,\mathbf{p}_N - \mathbf{p}_{i-1}\,)$$

$$= \mathbf{z}_{i-1} \times (\,\mathbf{p}_N - \mathbf{p}_{i-1}\,)\,\dot{\mathbf{q}}_i$$

Note that $\mathbf{o}_{i-1}$ has no translational velocity

origin of frame i-1 w/c contains $\mathbf{z}_{i-1}$ since joint is rotational

Computation Of End-Effector Velocity

For a translational joint i,

$$\mathbf{w}_i = 0$$

$$\mathbf{u}_i = \mathbf{z}_{i-1}\,\dot{\mathbf{q}}_i$$

The total velocity of the end-effector during coordinated motion is the superposition of all the elementary velocities that represent single joint motion:

$$\mathbf{v}_N = \begin{bmatrix} \mathbf{u}_N \\ \mathbf{w}_N \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} u_i \\ \sum_{i=1}^{N} w_i \end{bmatrix}$$

Computation Of End-Effector Velocity

$$\mathbf{v}_N = \underbrace{\begin{pmatrix} \overset{6x1}{J_1} & J_2 & J_3 & \ldots & J_N \end{pmatrix}}_{6xN\ J(q)} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_N \end{pmatrix}$$

Column J_i represents motion contribution of joint i

J(q) = Jacobian matrix
Cartesian $\leftrightarrow$ joint space

Computation Of End-Effector Velocity

For a translational joint i

$$\mathbf{J_i} = \begin{bmatrix} z_{i-1} \\ 0 \end{bmatrix}$$

For a rotational joint i

$$\mathbf{J_i} = \begin{bmatrix} z_{i-1} \times (p_N - p_{i-1}) \\ z_{i-1} \end{bmatrix}$$

Jacobian Transformations

- Velocities expressed in different frames

$$ {}^{A}\mathbf{v}_{N} \leftrightarrow {}^{B}\mathbf{v}_{N} \quad \begin{cases} N = \text{End Effector} \\ B = \text{may be a link coord frame that is held instantaneously constant} \end{cases} $$

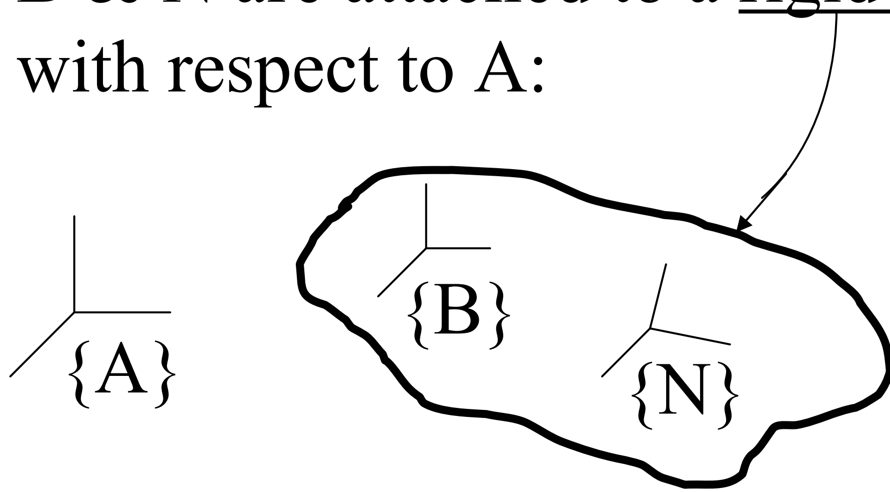
For ${}^{A}\mathbf{R}_{B}$ and ${}^{A}\mathbf{p}_{B}$ constants

$$ {}^{A}\mathbf{v}_{N} = \begin{pmatrix} {}^{A}\mathbf{u}_{N} \\ {}^{A}\mathbf{w}_{N} \end{pmatrix} = \underbrace{\left[\begin{array}{c:c} {}^{A}\mathbf{R}_{B} & 0 \\ \hdashline 0 & {}^{A}\mathbf{R}_{B} \end{array}\right]}_{\mathbf{J}} \underbrace{\begin{pmatrix} {}^{B}\mathbf{u}_{N} \\ {}^{B}\mathbf{w}_{N} \end{pmatrix}}_{{}^{B}\mathbf{v}_{N}} $$

Jacobian Transformations

- Diff pts on End-Effector

For $^B\mathbf{R}_N$ and $^B\mathbf{p}_N$ constants
B & N are attached to a rigid body moving
with respect to A:



N =, B = } Two Frames Attached to End-Effector

Jacobian Transformations

$$ {}^{A}\mathbf{u}_{N} = {}^{A}\mathbf{u}_{\mathbf{B}} + \lfloor {}^{A}\mathbf{w}_{\mathbf{B}} \mathrm{x} \rfloor ({}^{A}\mathbf{p}_{N} - {}^{A}\mathbf{p}_{\mathbf{B}}) $$

$$ {}^{A}\mathbf{w}_{N} = {}^{A}\mathbf{w}_{\mathbf{B}} $$

$$ {}^{A}\mathbf{v}_{N} = \begin{pmatrix} {}^{A}\mathbf{u}_{N} \\ {}^{A}\mathbf{w}_{N} \end{pmatrix} = \underbrace{\left[\begin{array}{c:c} \mathbf{I} & \lfloor -({}^{A}\mathbf{p}_{N} - {}^{A}\mathbf{p}_{\mathbf{B}}) \rfloor \mathrm{x} \\ \hdashline 0 & \mathbf{I} \end{array}\right]}_{\text{another } \mathbf{J}} \begin{bmatrix} {}^{A}\mathbf{u}_{\mathbf{B}} \\ {}^{A}\mathbf{w}_{\mathbf{B}} \end{bmatrix} $$

Jacobian Transformations

Most "simplified" form of Manipulator Jacobian is:

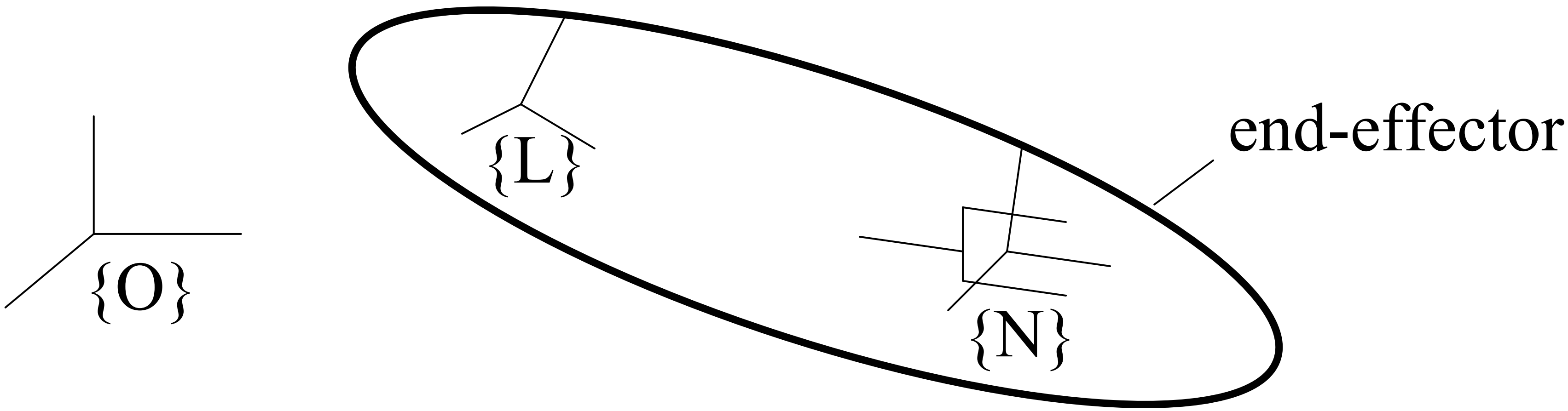
when it is computed at the mid-coord. frame

$$L \approx \frac{N+1}{2} \quad \text{, i.e.}$$

– All vectors are expressed in frame L

– The origin of Frame L is taken as the velocity pt for the end-effector

Jacobian Transformations



– Note that although L is moving, it is taken as instantaneously fixed when computing the Jacobian

Jacobian Transformations

- In computing the Jacobian ${}^{L}\mathbf{J}_{L}$, each column is

$$\mathbf{J}_i = \begin{pmatrix} {}^{L}\mathbf{z}_{i-1} \times (\mathbf{p}_L - \mathbf{p}_{i-1}) \\ {}^{L}\mathbf{z}_{i-1} \end{pmatrix} \text{ for rot. joint}$$

$$\mathbf{J}_i = \begin{pmatrix} {}^{L}\mathbf{z}_{i-1} \\ 0 \end{pmatrix} \text{ for translational joint}$$

Jacobian Transformations

- OR $\quad {}^{L}\mathbf{J}_{L}$ can be computed from ${}^{0}\mathbf{J}_{N}$:

$$\begin{pmatrix} {}^{L}\mathbf{u}_{L} \\ {}^{L}\mathbf{w}_{L} \end{pmatrix} = \begin{pmatrix} {}^{L}\mathbf{R}_{0} & 0 \\ 0 & {}^{L}\mathbf{R}_{0} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \lfloor -({}^{0}\mathbf{p}_{L} - {}^{0}\mathbf{p}_{N})\times \rfloor \\ 0 & \mathbf{I} \end{pmatrix} \begin{pmatrix} {}^{0}\mathbf{u}_{N} \\ {}^{0}\mathbf{w}_{N} \end{pmatrix}$$

$$\underbrace{}_{}\qquad\qquad\qquad\qquad \underbrace{}_{}$$

$${}^{L}\mathbf{J}_{L}\,\dot{\mathbf{q}} = \Big[\qquad \text{''} \qquad \Big]\, {}^{0}\mathbf{J}_{N}\,\dot{\mathbf{q}}$$

$${}^{L}\mathbf{J}_{L} = \Big[\qquad \text{''} \qquad \Big]\, {}^{0}\mathbf{J}_{N}$$

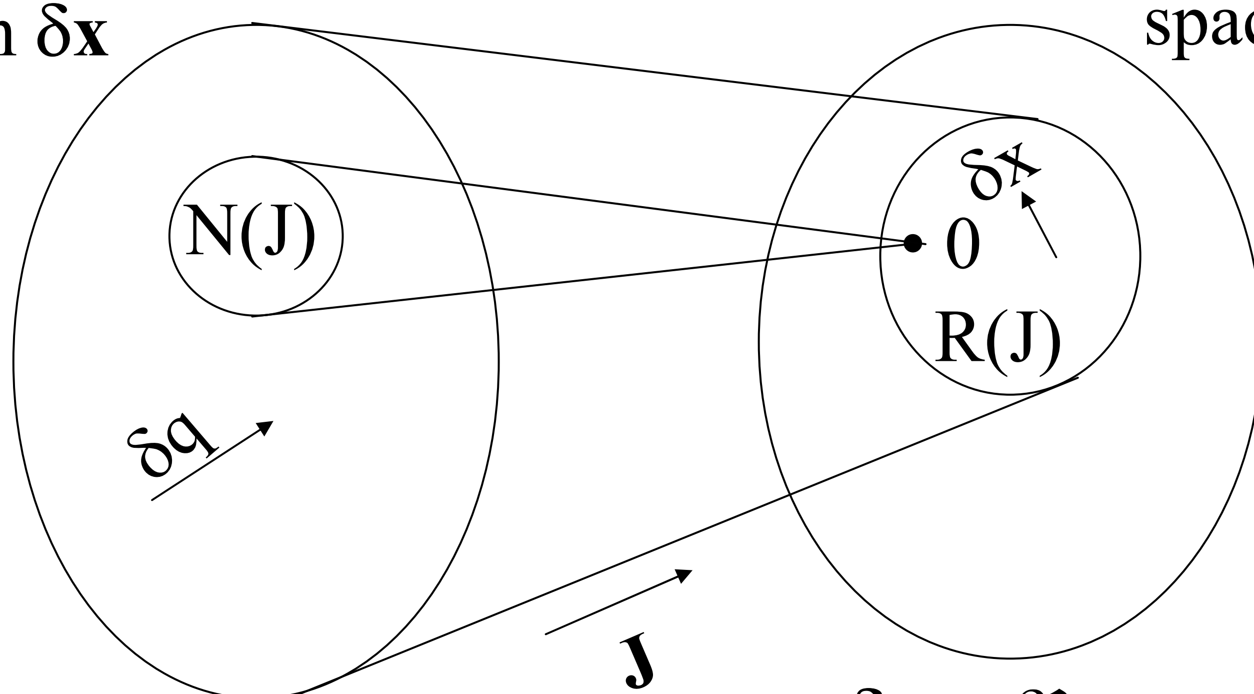
Robot Kinematics of Velocity

N(J) = Null space of J

$\delta\mathbf{q} \rightarrow$ produces no motion $\delta\mathbf{x}$

R(J) = Range Space of **J**

= or column space of **J**



$\delta\mathbf{q} \in \Re^{\mathbf{n}}$
Joint Space

$\delta\mathbf{x} \in \Re^{\mathbf{m}}$
End-Effector Space

Robot Kinematics of Velocity

$\mathbf{x} = \mathbf{G}(\mathbf{q})$

$\dot{\mathbf{x}} = \mathbf{J}\,\dot{\mathbf{q}}$

$\delta\mathbf{x} = \mathbf{J}\,\delta\mathbf{q}$

$\mathbf{J}$ = Jacobian

$$\mathbf{J}_{\mathbf{ij}} = \frac{\delta}{\delta q_j}\mathbf{G}_{\mathbf{i}}(\mathbf{q})$$

$\mathbf{x}$ = representation for E-E configuration

$$\dot{\mathbf{x}} = \begin{pmatrix} E_p(x_p) & 0 \\ 0 & E_r(x_r) \end{pmatrix} \underbrace{\begin{pmatrix} u \\ w \end{pmatrix}}_{} \qquad \dot{\mathbf{x}} = \begin{pmatrix} \quad \end{pmatrix} \mathbf{v}$$

$\mathbf{v}$ = E - ε – Angular velocity

$\mathbf{v} = \mathbf{J_0}\,\dot{\mathbf{q}}$ = Basic Jacobian

Robot Kinematics of Velocity

$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} E_p(x_p) & 0 \\ 0 & E_r(x_r) \end{pmatrix} \mathbf{J}_0(\mathbf{q})}_{} \, \dot{\mathbf{q}}$$

$$\mathbf{J} = \text{Jacobian} = \mathbf{E}\,\mathbf{J}_0(\mathbf{q})$$

$$\delta\mathbf{x} = \mathbf{J}\,\delta\mathbf{q}$$

Inverse Kinematics of Velocity

Solution to $\delta\mathbf{x} = \mathbf{J}\,\delta\mathbf{q}$ [i.e., given $\delta\mathbf{x}$, Find $\delta\mathbf{q}$]

$\delta\mathbf{x}$: mx1 $\delta\mathbf{q}$: nx1

Exists if & only if

Rank $\mathbf{J}$ = Rank ($\mathbf{J}$ | $\delta\mathbf{x}$)

$\mathbf{J}$: m x n

($\mathbf{J}$ | $\delta\mathbf{x}$): m x (n + 1) matrix obtained by augmenting $\mathbf{J}$ with column $\delta\mathbf{x}$

Meaning $\delta\mathbf{x}$ must be in the subspace spanned by the columns of $\mathbf{J}$

Inverse Kinematics of Velocity

First: Convert $\delta \mathbf{x}$ to $\delta \mathbf{x_0} \in \mathbf{R}^{\mathbf{m_o}}$ (velocity, basic kinematic model)

$$\delta \mathbf{x_0} = \mathbf{J_0}\, \mathbf{c} \qquad ({}^{0}\dot{\mathbf{v}}_{\mathrm{N}} = \mathbf{J_0}\, \dot{\mathbf{q}})$$

$\delta \mathbf{x_0} \rightarrow \mathbf{R}^{\mathbf{m_o}}$, $\quad \mathbf{c} \rightarrow \mathbf{R}^{\mathbf{n}}$ $\qquad m_0 \leq 6$

Solution exists if & only if <u>Rank $\mathbf{J_0} = \min(m_0, n)$</u>

i.e., columns of $\mathbf{J_0}$ span the space $\mathbf{R}^{\mathbf{\min(m_o, n)}}$

Inverse Kinematics of Velocity

General Solution:

generalized Inverse

$$\delta \mathbf{q} = \mathbf{J_0}^{\#}(\mathbf{q})\ \delta \mathbf{x_0} + [\ \mathbf{I_n} - \mathbf{J_0}^{\#}(\mathbf{q})\ \mathbf{J_0}(\mathbf{q})\ ]\ \delta \mathbf{q_0}$$

nxn Identity

any arbitrary disp

Operates on $\delta \mathbf{q_0}$ to produce vector $\delta \mathbf{q_n} \in \mathbf{N(J)}$

$$\delta \mathbf{q_n} = [\ \mathbf{I_n} - \mathbf{J_0}^{\#}(\mathbf{q})\ \mathbf{J_0}(\mathbf{q})\ ]\ \delta \mathbf{q_0}$$

The mapping by $\mathbf{J_0}$ of $\delta \mathbf{q_n}$ results in zero vector in $\mathbf{R}^{\mathbf{m_0}}$

$$\mathbf{J_0}\ \delta \mathbf{q_n} = [\mathbf{J_0} - \mathbf{J_0}\ \mathbf{J_0}^{\#}(\mathbf{q})\ \mathbf{J_0}(\mathbf{q})\ ]\ \delta \mathbf{q_0} = 0$$

Inverse Kinematics of Velocity

Case 1: $m_0 = n \quad \leq 6$

$\mathbf{J}^{\#}(\mathbf{q}) = \mathbf{J}^{-1}$ (possible problem with singularity, $\mathbf{J}^{-1}$ may not exist)

Case 2: $m_0 > n, \quad m_0 \leq 6$ (not interesting/useful case, task shall be $\leq n$)

overdetermined system: more eqns than unknowns.

$\mathbf{J}^{\#} = (\mathbf{J}^T \mathbf{J}^{-1})\, \mathbf{J}^T$ = left pseudo inverse

= exists only if Rank $\mathbf{J} = n$

Sol'n minimizes $\| \mathbf{J}\delta\mathbf{q} - \delta\mathbf{x_0} \|_2$

Inverse Kinematics of Velocity

Case 3: $m_0 < n$, $m_0 \leq 6$ (Redundant Robots)

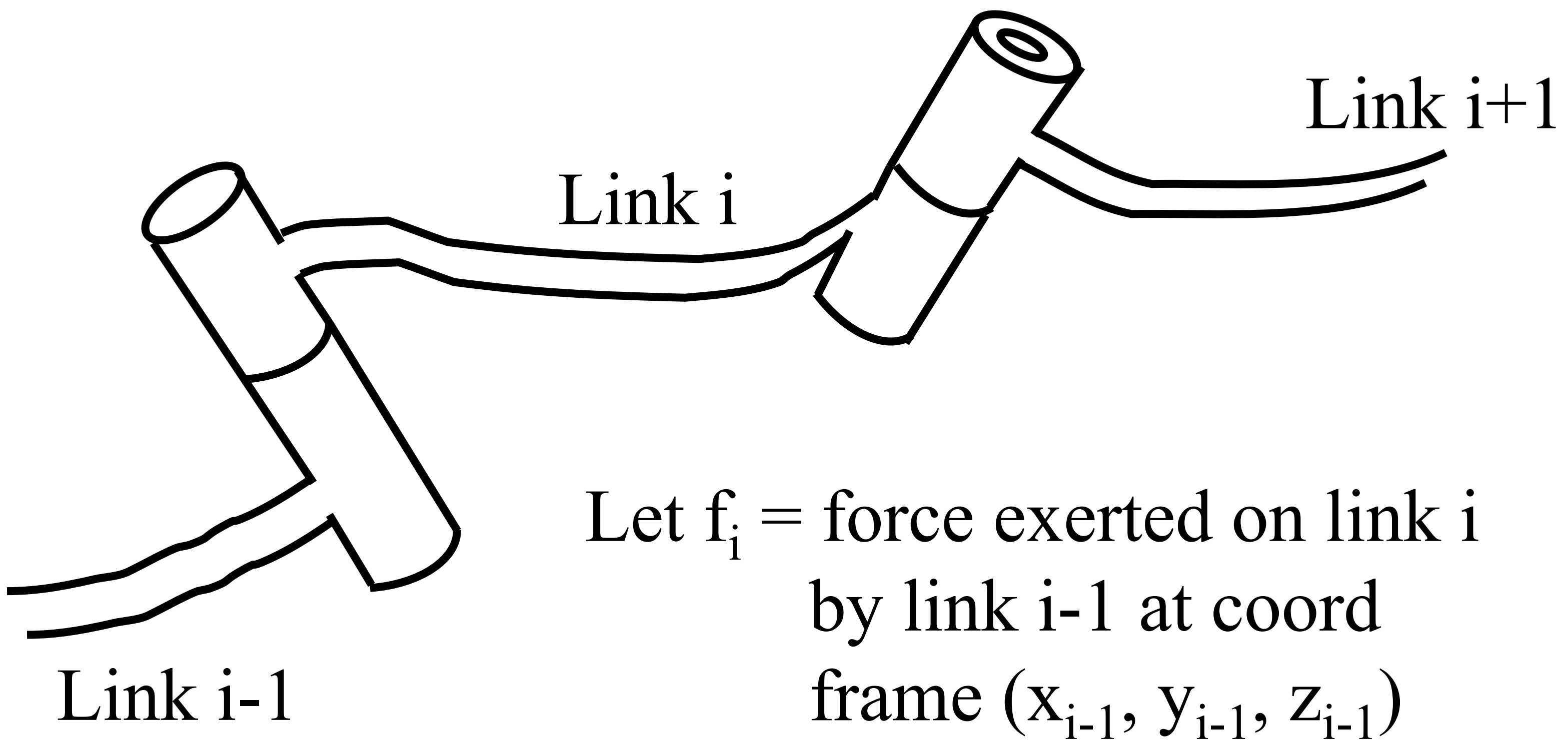
underdetermined system = less eqns than unknowns

$\mathbf{J}^{\#} = \mathbf{J}^T (\mathbf{J}\,\mathbf{J}^T)^{-1}$ = right pseudo inverse

= exists only if Rank $\mathbf{J} = m_0$

Sol'n minimizes $\| \delta\mathbf{q} \|_2$

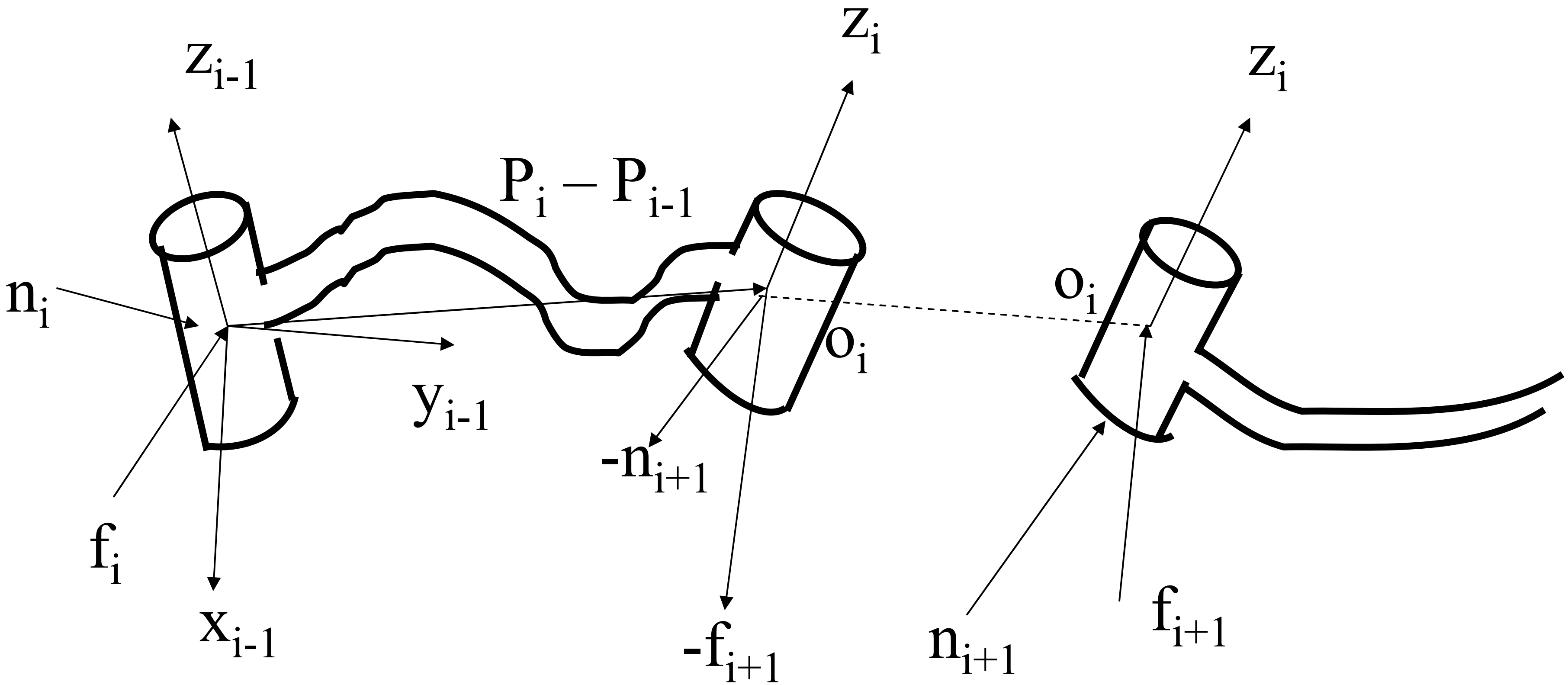
Static Forces in Manipulators



Let f_i = force exerted on link i by link i-1 at coord frame $(x_{i-1}, y_{i-1}, z_{i-1})$

n_i = moment exerted on link i

Static Forces in Manipulators



Static Forces in Manipulators

$$\left\{\begin{array}{l}\sum \mathbf{F} = 0 \quad \mathbf{f_i} - \mathbf{f_{i+1}} = 0 \\ \sum \text{Torques about origin of frame i-1} = 0 \\ \quad \mathbf{n_i} - \mathbf{n_{i+1}} + (\mathbf{p_i} - \mathbf{p_{i-1}}) \times (-\mathbf{f_{i+1}}) = 0\end{array}\right.$$

If we start with a description of the force and moment applied by the hand, we can calculate the force and moment applied by each link working from the last link down to the base, link ϕ.

$$\left.\begin{array}{l}\mathbf{f_{n+1}} \\ \mathbf{n_{n+1}}\end{array}\right\}$$ Force exerted by the manipulator hand on its environment.

Static Forces in Manipulators

Recursive Equations:

$$\mathbf{f}_i = \mathbf{f}_{i+1}$$
$$\mathbf{n}_i = \mathbf{n}_{i+1} + (\mathbf{p}_i - \mathbf{p}_{i-1}) \times \mathbf{f}_{i+1}$$

all vectors expressed in same frame (e.g. base frame ϕ)

What forces are Needed at the Joints in order to Balance the Reaction Forces & Moments acting in the link

$$\mathbf{T}_i = \begin{cases} \mathbf{n}_i^T \mathbf{z}_{i-1} & \text{for a rotational link } i \\ \mathbf{f}_i^T \mathbf{z}_{i-1} & \text{for a translational link } i \end{cases}$$

Jacobians In Force Domain

- When forces act on a mechanism, work (in the technical sense) is done if the mechanism moves through a displacement
- Principle if VIRTUAL WORK allows us to make certain statements about the static case by defining a VIRTUAL DISPLACEMENT δx that is experienced without passage of time $dt = 0$
  (Not only infinitesimal, $dx \neq \delta x$)

Jacobians In Force Domain

- Since work has units of energy, it must be the same measured in any set of generalized coordinates

$$\mathbf{F} \cdot \delta\mathbf{x} = \tau \cdot \delta\mathbf{q}$$

- $\mathbf{F}$: 6x1 Cartesian Force-Moment Vector
- $\delta\mathbf{x}$: 6x1 Virtual displ. In Cartesian space
- τ: Torque/Forces at joints 6x1
- $\delta\mathbf{q}$: 6x1 virtual joint disp.

Jacobians In Force Domain

- But $\delta \mathbf{x} = \mathbf{J}\, \delta \mathbf{q}$

- Therefore $\mathbf{F}^{\mathbf{T}} \underbrace{[\mathbf{J}\, \delta \mathbf{q}]}_{\delta \mathbf{x}} = \underline{\tau^{\mathbf{T}}\, \delta \mathbf{q}}$

$$\mathbf{F}^{\mathbf{T}}\, \mathbf{J} = \tau^{\mathbf{T}}$$

$$\boxed{\tau = \mathbf{J}^{\mathbf{T}}\, \mathbf{F}}$$

expressed in the same (<u>consistent</u>) Frame

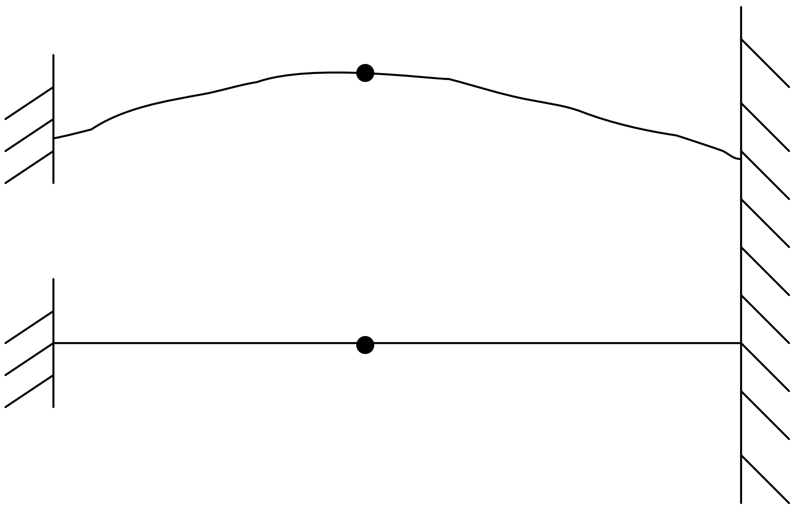
Jacobians In Force Domain

- When the Jacobian loses full rank, there are certain directions in which the end-effector cannot exert static forces (through joint actuation) as desired

- That is, if **J** is singular, the equation is not valid
  - **F** could be increased or decreased in certain directions with no effect on the value calculated for τ
  - These directions are in the null-space of the Jacobian

Jacobians In Force Domain

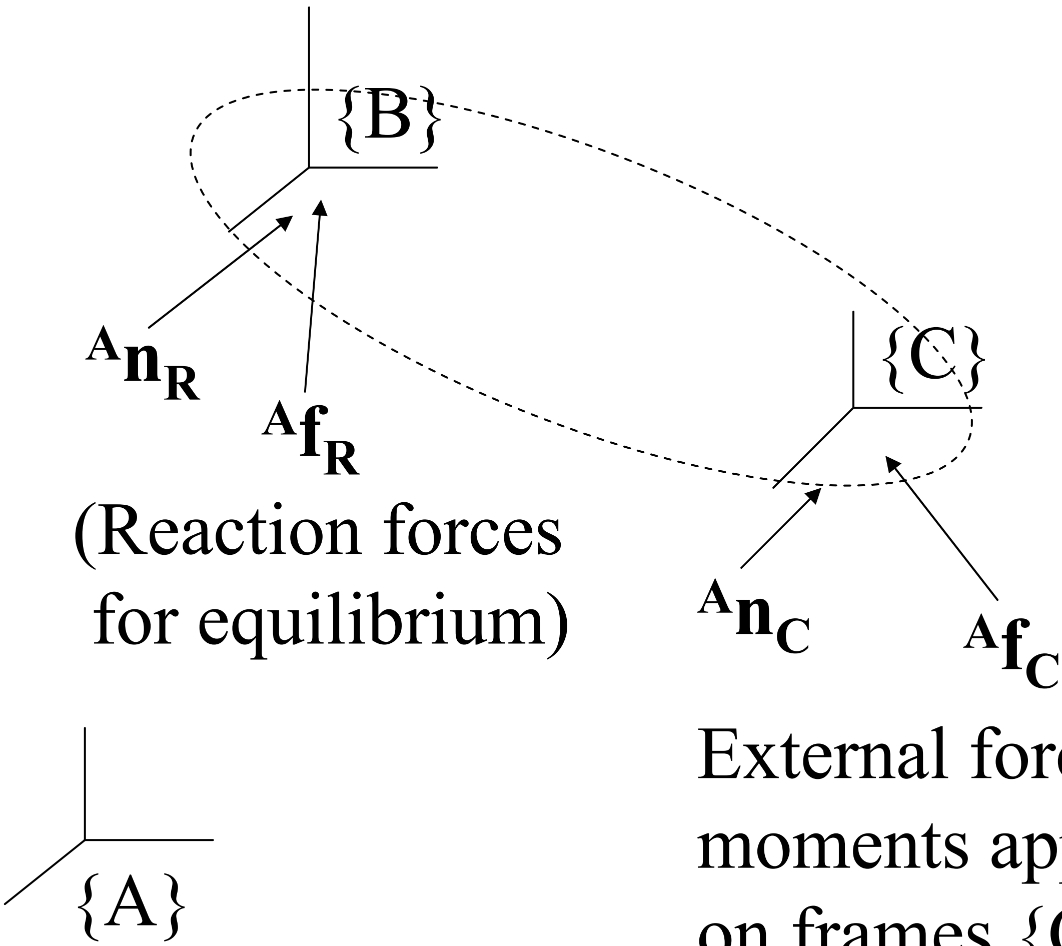
- This also means that near singular configuration, mechanical advantage tends towards infinity, such that with small joint torques, large forces could be generated at the end-effector



Jacobians In Force Domain

- Note that a Cartesian space quantity can be converted into a joint space quantity without calculating any inverse kinematic functions.

Cartesian Transformation Of Static Force



Given: $^A f_C$

$^A n_C$

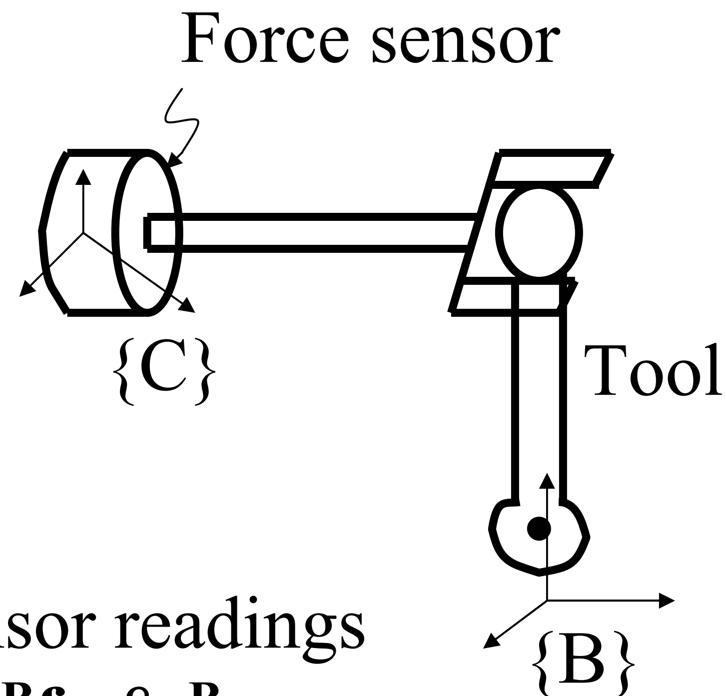
Find: $^A f_B$

$^A n_B$

(the force/moment experienced at B if force/moment is exerted on C)

Cartesian Transformation Of Static Force

Why is this important?



$^{C}\mathbf{f}_{C}$ & $^{C}\mathbf{n}_{C}$ can be force sensor readings
But our primary interest is $^{B}\mathbf{f}_{B}$ & $^{B}\mathbf{n}_{B}$
(force/moments at tool tip)

Cartesian Transformation Of Static Force

Equilibrium:

$\sum F = 0$ $\qquad$ ${}^{A}f_{C} + {}^{A}f_{R} = 0$

$${}^{A}f_{R} = - {}^{A}f_{C}$$

$\sum N = 0$ $\qquad$ ${}^{A}n_{C} + ({}^{A}p_{B} - {}^{A}p_{C}) \times {}^{A}f_{R} + n_{R} = 0$

$${}^{A}n_{R} = - {}^{A}n_{C} - ({}^{A}p_{B} - {}^{A}p_{C}) \times {}^{A}f_{R}$$

But ${}^{A}f_{B} = - {}^{A}f_{R} \rightarrow$

$$\boxed{{}^{A}f_{B} = {}^{A}f_{C}}$$

Cartesian Transformation Of Static Force

$$ {}^{A}\mathbf{n}_{B} = -{}^{A}\mathbf{n}_{R} = {}^{A}\mathbf{n}_{C} + ({}^{A}\mathbf{p}_{B} - {}^{A}\mathbf{p}_{C}) \times {}^{A}\mathbf{f}_{R} $$

$$ {}^{A}\mathbf{n}_{B} = {}^{A}\mathbf{n}_{C} + ({}^{A}\mathbf{p}_{B} - {}^{A}\mathbf{p}_{C}) \times (-{}^{A}\mathbf{f}_{C}) $$

$$ \boxed{{}^{A}\mathbf{n}_{B} = {}^{A}\mathbf{n}_{C} + ({}^{A}\mathbf{p}_{C} - {}^{A}\mathbf{p}_{B}) \times {}^{A}\mathbf{f}_{C}} $$

OR

$$ {}^{A}\mathbf{n}_{B} = {}^{A}\mathbf{n}_{C} + \lfloor {}^{A}\mathbf{R}_{B}\, {}^{B}\mathbf{p}_{C} \mathbf{x} \rfloor\ {}^{A}\mathbf{f}_{C} $$

in Matrix Form

$$ \begin{bmatrix} {}^{A}\mathbf{f}_{B} \\ \hdashline {}^{A}\mathbf{n}_{B} \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} \mathbf{I} & 0 \\ \hline \lfloor {}^{A}\mathbf{R}_{B}\, {}^{B}\mathbf{p}_{C} \mathbf{x} \rfloor & \mathbf{I} \end{array}\right]}_{\text{Force torque Jacobian transformation}} \begin{bmatrix} {}^{A}\mathbf{f}_{C} \\ \hdashline {}^{A}\mathbf{n}_{C} \end{bmatrix} $$

Force torque Jacobian transformation

Cartesian Transformation Of Static Force

But in typical applications, we would like to relate

$$\begin{bmatrix} {}^{C}\mathbf{f}_{C} \\ {}^{C}\mathbf{n}_{C} \end{bmatrix} \text{with} \begin{bmatrix} {}^{B}\mathbf{f}_{B} \\ {}^{B}\mathbf{n}_{B} \end{bmatrix}$$

[e.g. sensor readings will be expressed in local frame of sensor]

We can transform vectors **f** & **n** like any other vector via Rotation Matrices

$$\begin{bmatrix} {}^{A}\mathbf{f}_{C} \\ {}^{A}\mathbf{n}_{C} \end{bmatrix} = \left[\begin{array}{c:c} {}^{A}\mathbf{R}_{C} & 0 \\ \hdashline 0 & {}^{A}\mathbf{R}_{C} \end{array}\right] \begin{bmatrix} {}^{C}\mathbf{f}_{C} \\ {}^{C}\mathbf{n}_{C} \end{bmatrix}$$

Cartesian Transformation Of Static Force

$\therefore$

$$\begin{bmatrix} {}^{A}\mathbf{f}_{B} \\ {}^{A}\mathbf{n}_{B} \end{bmatrix} = \left[\begin{array}{c:c} \mathbf{I} & 0 \\ \hdashline \lfloor {}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{p}_{C}\mathbf{x} \rfloor & \mathbf{I} \end{array}\right] \left[\begin{array}{c:c} {}^{A}\mathbf{R}_{C} & 0 \\ \hdashline 0 & {}^{A}\mathbf{R}_{C} \end{array}\right] \begin{bmatrix} {}^{C}\mathbf{f}_{C} \\ {}^{C}\mathbf{n}_{C} \end{bmatrix}$$

$$\begin{bmatrix} {}^{A}\mathbf{f}_{B} \\ {}^{A}\mathbf{n}_{B} \end{bmatrix} = \left[\begin{array}{c:c} {}^{A}\mathbf{R}_{C} & 0 \\ \hdashline \lfloor {}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{p}_{C}\mathbf{x} \rfloor\,{}^{A}\mathbf{R}_{C} & {}^{A}\mathbf{R}_{C} \end{array}\right] \begin{bmatrix} {}^{C}\mathbf{f}_{C} \\ {}^{C}\mathbf{n}_{C} \end{bmatrix}$$

Also

$$\begin{bmatrix} {}^{B}\mathbf{f}_{B} \\ {}^{B}\mathbf{n}_{B} \end{bmatrix} = \left[\begin{array}{c:c} {}^{B}\mathbf{R}_{A} & 0 \\ \hdashline 0 & {}^{B}\mathbf{R}_{A} \end{array}\right] \begin{bmatrix} {}^{A}\mathbf{f}_{B} \\ {}^{A}\mathbf{n}_{B} \end{bmatrix}$$

Cartesian Transformation Of Static Force

Therefore

$$\begin{bmatrix} {}^{B}\mathbf{f}_{B} \\ \hdashline {}^{B}\mathbf{n}_{B} \end{bmatrix} = \begin{bmatrix} {}^{B}\mathbf{R}_{A} & 0 \\ 0 & {}^{B}\mathbf{R}_{A} \end{bmatrix} \begin{bmatrix} {}^{A}\mathbf{R}_{C} & 0 \\ \lfloor {}^{A}\mathbf{R}_{B}\, {}^{B}\mathbf{p}_{C} \times \rfloor\, {}^{A}\mathbf{R}_{C} & {}^{A}\mathbf{R}_{C} \end{bmatrix} \begin{bmatrix} {}^{C}\mathbf{f}_{C} \\ \hdashline {}^{C}\mathbf{n}_{C} \end{bmatrix}$$

$$= \begin{bmatrix} {}^{B}\mathbf{R}_{A}\, {}^{A}\mathbf{R}_{C} & 0 \\ {}^{B}\mathbf{R}_{A} \lfloor {}^{A}\mathbf{R}_{B}\, {}^{B}\mathbf{p}_{C} \times \rfloor\, {}^{A}\mathbf{R}_{C} & {}^{B}\mathbf{R}_{A}\, {}^{A}\mathbf{R}_{C} \end{bmatrix} \begin{bmatrix} {}^{C}\mathbf{f}_{C} \\ {}^{C}\mathbf{n}_{C} \end{bmatrix}$$

$$({}^{B}\mathbf{R}_{A} \lfloor {}^{A}\mathbf{R}_{B}\, {}^{B}\mathbf{p}_{C} \times \rfloor\, {}^{A}\mathbf{R}_{C})\, {}^{C}\mathbf{f}_{C} = {}^{B}\mathbf{R}_{A} [({}^{A}\mathbf{R}_{B}\, {}^{B}\mathbf{p}_{C}) \times ({}^{A}\mathbf{R}_{C}\, {}^{C}\mathbf{f}_{C})]$$

$$= ({}^{B}\mathbf{R}_{A}\, {}^{A}\mathbf{R}_{B}\, {}^{B}\mathbf{p}_{C}) \times ({}^{B}\mathbf{R}_{A}\, {}^{A}\mathbf{R}_{C}\, {}^{C}\mathbf{f}_{C})$$

$$= {}^{B}\mathbf{p}_{C} \times ({}^{B}\mathbf{R}_{C}\, {}^{C}\mathbf{f}_{C})$$

$$= \lfloor {}^{B}\mathbf{p}_{C} \times \rfloor\, {}^{B}\mathbf{R}_{C}\, {}^{C}\mathbf{f}_{C}$$

Cartesian Transformation Of Static Force

$$\begin{bmatrix} {}^{B}\mathbf{f}_{B} \\ {}^{B}\mathbf{n}_{B} \end{bmatrix} = \left[\begin{array}{c:c} {}^{B}\mathbf{R}_{C} & 0 \\ \hdashline \lfloor {}^{B}\mathbf{p}_{C}\mathbf{x} \rfloor {}^{B}\mathbf{R}_{C} & {}^{B}\mathbf{R}_{C} \end{array}\right] \begin{bmatrix} {}^{C}\mathbf{f}_{C} \\ {}^{C}\mathbf{n}_{C} \end{bmatrix}$$

→ This is the form given in Craig's Book