CHAPTER 3

Robot Trajectory Planning & Kinematic Robot Control

After this chapter, the students are expected to learn the following:

1. Plan point to point trajectories in joint space and task space

2. Plan trajectories with via points

3. Plan trajectories with velocity and acceleration constraints

4. Kinematic control algorithm using resolved motion rate control
Trajectory Generation

- Polynomial Trajectory
  - subject to constraints
  - boundary conditions

- Initial & Final Positions

\[
\begin{align*}
\theta(0) &= \theta_0 \\
\theta(t_f) &= \theta_f
\end{align*}
\]

Initial & Final Velocities

\[
\begin{align*}
\dot{\theta}(0) &= 0 \\
\dot{\theta}(t_f) &= 0
\end{align*}
\]

4 constraints

Cubic Polynomial

\[
\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3
\]

Four Equations (constraints) in Four Unknowns \((a_0, \ldots, a_3)\)

\[
\begin{align*}
a_0 &= \theta_0 \\
a_2 &= \frac{3}{t_f^2} (\theta_f - \theta_0) \\
a_1 &= 0 \\
a_3 &= \frac{-2}{t_f^3} (\theta_f - \theta_0)
\end{align*}
\]
Trajectory Generation

- Cubic Polynomials with Via Points

- Choose velocities at via points

Quintic Polynomial
\[ \theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 \]

6 Equations in 6 unknowns:

- \( a_0 = \theta_0 \)
- \( a_1 = \theta_0 \)
- \( a_2 = \frac{\theta_0}{2} \)
Trajectory Generation

$$a_3 = \frac{20 \theta_f - 20 \theta_0 - (\delta \dot{\theta}_f + 12 \dot{\theta}_0) t_f - (3 \ddot{\theta}_0 - \ddot{\theta}_f) t_f^2}{2 t_f^3}$$

$$a_4 = \frac{30 \theta_0 - 30 \theta_f + (14 \dot{\theta}_f + 16 \dot{\theta}_0) t_f + (3 \ddot{\theta}_0 - 2 \ddot{\theta}_f) t_f^2}{2 t_f^4}$$

$$a_5 = \frac{12 \theta_f - 12 \theta_0 - (6 \dot{\theta}_f + 6 \dot{\theta}_0) t_f - (\ddot{\theta}_0 - \ddot{\theta}_f) t_f^2}{2 t_f^5}$$

Trajectory Generation

- Linear Segment with Parabolic Blend

- constant acceleration

- constant velocity

- constant deceleration

- same magnitude of acceleration

- same Δt’s at start & end

- same magnitude of slope

- Velocity profile

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Trajectory Generation

There are many possible solutions, given
\[ \theta(0) = \theta_0 \quad \theta(0) = 0 \]
\[ \theta(t_f) = \theta_f \quad \dot{\theta}(t_f) = 0 \]

\[ \theta_b = \theta_0 + \frac{1}{2} \ddot{\theta} t_b^2 \quad \text{— (1)} \]
\[ \dot{\theta}_b = \dot{\theta}_0 + \ddot{\theta}_b = \frac{\theta_h - \theta_b}{t_h - t_b} \quad \text{— (2)} \]

where \( \theta_h = \frac{1}{2} \left( \frac{\theta_f + \theta_0}{2} \right) \)

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Trajectory Generation

Given \( \theta_0, \theta_f \) and \( t_f \) (move time)

Can choose \( \ddot{\theta} \) and \( t_b \) to satisfy (1) & (2)

Combining (1) & (2) and since \( t_f = 2t_h \), we get

\[ \ddot{\theta} t_b^2 - \dot{\theta}_f t_b + (\theta_f - \theta_0) = 0 \]

\[ t_b = \frac{t_f}{2} - \frac{\sqrt{\ddot{\theta} t_f^2} - 4\ddot{\theta} (\theta_f - \theta_0)}{2\ddot{\theta}} \]
Trajectory Generation

For $t_b$ to exists,

$$\ddot{\theta} \geq \frac{4(\theta_f - \theta_o)}{t_f^2}$$

(acceleration must be sufficiently high)

When equality holds:

$$t_b = \frac{t_f}{2} = t_h$$

(no constant velocity)

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Trajectory Generation

- Linear Segments with Parabolic Blods & with Via Points

given: - all points $\theta_k$
- all duration $t_{dk}$
- magnitude of acceleration $|\theta_k|$

compute: bold times $t_k$

duration between $\theta_j$ & $\theta_k$
Trajectory Generation

Interior Paths:

\[ \dot{\theta}_{jk} = \frac{\theta_k - \theta_j}{t_{dk}} \]
\[ \dot{\theta}_{kl} = \frac{\theta_l - \theta_k}{t_{dl}} \]
\[ \ddot{\theta}_k = SGN(\dot{\theta}_{kl} - \dot{\theta}_{jk}) |\ddot{\theta}_k| \]
\[ t_k = \frac{\dot{\theta}_{kl} - \dot{\theta}_{jk}}{\ddot{\theta}_k} \]
\[ t_{jk} = t_{dk} - \frac{1}{2} t_j - \frac{1}{2} t_k \]

Trajectory Generation

First Segment (difficult because entire block region crowded in duration)

\[ \theta_2 - \theta_1 = \ddot{\theta}_1 t_1 = \text{velocity at 1st line segment} = \dot{\theta}_{12} \]
\[ t_{d_{12}} - \frac{1}{2} t_1 \]
where \( \ddot{\theta}_1 = SGN(\theta_2 - \theta_1) |\ddot{\theta}_1| \)

\[ t_1 = t_{d_{12}} - \sqrt{t_{d_{12}}^2 - \frac{2(\theta_2 - \theta_1)}{\dot{\theta}_1}} \]
\[ \dot{\theta}_{12} = \frac{\theta_2 - \theta_1}{t_{d_{12}} - \frac{1}{2} t_1} \]
\[ t_{12} = t_{d_{12}} - t_1 - \frac{1}{2} t_2 \]
Trajectory Generation

Last segment connecting $\theta_{n-1}$ & $\theta_n$

\[
\frac{\theta_n - \theta_{n-1}}{t_d_{(n-1)n} - \frac{1}{2} t_n} = -\ddot{\theta}_n t_n
\]

where $\ddot{\theta}_n = \text{SGN}(\theta_n - \theta_{n-1}) |\ddot{\theta}_n|$

\[
t_n = t_{d_{(n-1)n}} - \sqrt{t_{d_{(n-1)n}}^2 + \frac{2(\theta_n - \theta_{n-1})}{\dot{\theta}_n}}
\]

\[
\dot{\theta}_{(n-1)n} = \frac{\theta_n - \theta_{n-1}}{t_{d_{(n-1)n}} - \frac{1}{2} t_n}
\]

\[
t_{d_{(n-1)n}} = t_n - \frac{1}{2} t_{n-1}
\]

Resolved Motion Rate Control

– Kinematic Control without the need for solving the inverse Kinematics of Position

– Need Joint level control which is available in robot controllers

\[
\delta q \quad \delta x
\]

– Command joint motion such that desired e-e motion is achieved
Resolved Motion Rate Control

1) Given a Trajectory $x(t) \in \mathbb{R}^m$ in task space

![Initial position](initial_position) \rightarrow f

2) Divide Trajectory into small segments according to sample time on reference Trajectory update rate

3) At $x_k$, compute $E(x_k)$, compute $J_0(q_k)$

4) Compute $\Delta x_k = x_{k+1} - x_k$

5) Compute $\delta x_{0,k} = E(x_k) + \Delta x_k$

6) Compute $\delta q = J_0(q_k)^\# \delta x_{0,k} + \left[ \text{In} - J_0(q_k)^\# J_0(q_k) \right] \delta q_0$

7) Command $\delta q$ to robot controller (Robot moves from $q_k$ to $q_{k+1}$) ($\delta q = q_{k+1} - q_k$)

8) Go to step 3 until $x_k$ reaches $x_f$