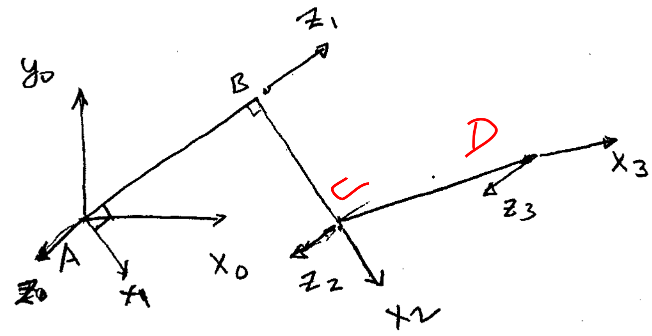
Sol^n. for ME4245/EE4304 – Robotics
By Marcelo Ang & Prahlad

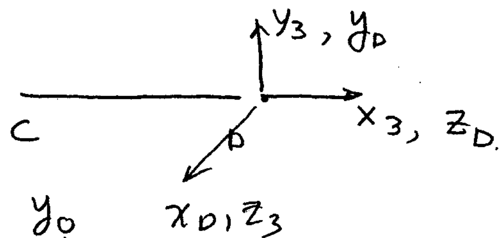
Q1 (a)

(i)

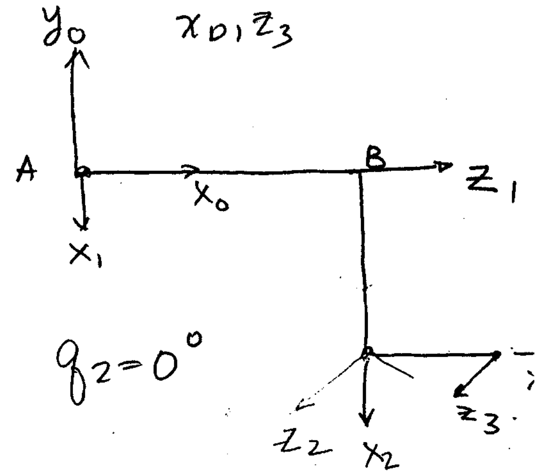


z_0, z_2, z_3 } out of paper

Last Link



at $q_1 = -90^\circ$



$O_0 = O_1$

(ii)

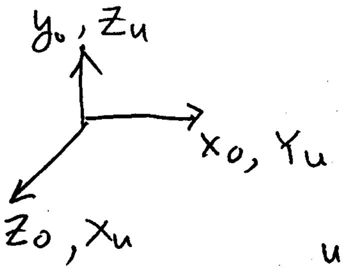
	θ	r	d	α
1	$q_1 =$	0	0 $\cancel{1}$	-90°
2	$q_2 =$	1	1	90°
3	q_3	0	1	0°

(iii) $q_1 = \phi_1 + (-90^\circ)$

$q_2 = \phi_2$

$q_3 = \phi_3$

(iv). Frame 0 & U.



$${}^{u}T_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad {}^{3}T_D = \begin{pmatrix} 0 & 0 & 1 & t \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$${}^{u}T_D = {}^{u}T_0 \, {}^{0}T_1 \, {}^{1}T_2 \, {}^{2}T_3 \, {}^{3}T_D.$$

$${}^{u}z_D = {}^{u}T_D \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad {}^{u}p_D = {}^{u}T_D \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

${}^{0}T_1 =$ ____, ${}^{1}T_2 =$ ${}^{2}T_3 =$

Q1.

(a)

$${}^{A}T_{B_0} = \text{initial position of } B = {}^{A}T_{B} = \text{given}$$

$${}^{A}T_{B_1} = Rot(z, 30^\circ)\, {}^{A}T_{B_0}$$

$${}^{A}T_{C_1} = {}^{A}T_{B_1}\, {}^{B_1}T_{C_1} \quad \text{where } {}^{B_1}T_{C_1} = {}^{B}T_{C} = \text{given}$$

$${}^{A}T_{C_2} = {}^{A}T_{C_1}\, Rot(x, 40^\circ)$$

$${}^{D}T_{C_2} = {}^{D}T_{A}\, {}^{A}T_{C_2} \quad \text{where } {}^{D}T_{A} = {}^{A}T_{D}^{-1} = \text{given}$$

$${}^{D}T_{C_3} = Rot(y, 50^\circ)\, {}^{D}T_{C_2}$$

$${}^{D}T_{B_3} = {}^{D}T_{C_3}\, {}^{C_3}T_{B_3} \quad \text{where } {}^{B_3}T_{C_3} = \text{given} = {}^{B}T_{C}$$

$${}^{A}T_{B_3} = {}^{A}T_{D}\, {}^{D}T_{B_3} \quad //$$

or

$${}^{A}T_{B_3} = {}^{A}T_{D}\, Rot(y, 50^\circ)\, {}^{D}T_{C_2}\, {}^{C}T_{B}$$

$$= {}^{A}T_{D}\, Rot(y, 50^\circ)\, {}^{D}T_{A}\, {}^{A}T_{C_2}\, {}^{C}T_{B}.$$

$$= {}^{A}T_{D}\, Rot(y, 50^\circ)\, {}^{D}T_{A}\, {}^{A}T_{C_1}\, Rot(x, 40^\circ)\, {}^{C}T_{B}$$

$$= \underbrace{{}^{A}T_{D}\, Rot(y, 50^\circ)\, \underbrace{{}^{D}T_{A}\, \underbrace{\underbrace{\underbrace{Rot(z, 30^\circ)\, {}^{A}T_{B_0}}_{{}^{A}T_{B_1}}\, {}^{B}T_{C}}_{{}^{A}T_{C_1}}\, Rot(x, 40^\circ)}_{{}^{A}T_{C_2}}}_{{}^{D}T_{C_2}}\, {}^{C}T_{B}}_{{}^{A}T}$$

Q2 (a)

For $\sin\beta = n_x \neq 0$

$$O_x = \sin\beta \sin\gamma \ , \quad a_x = \sin\beta \cos\gamma$$

$$\gamma = \text{ATAN2}(O_x, a_x) \quad \text{if } \sin\beta > 0$$
$$= \text{ATAN2}(-O_x, -a_x) \quad \text{if } \sin\beta < 0$$

$$\alpha = \text{ATAN2}(n_y, -n_z) \quad \text{if } \sin\beta > 0$$
$$= \text{ATAN2}(-n_y, n_z) \quad \text{if } \sin\beta < 0$$

$$\beta = \text{ATAN2}(\pm\sqrt{1-n_x^2}, n_x)$$

2 sol'ns

$\beta = \text{ATAN2}(+\sqrt{1-n_x^2}, n_x)$
$\alpha = \text{ATAN2}(n_y, -n_z)$
$\gamma = \text{ATAN2}(O_x, a_x)$

$\beta = -\text{ATAN2}(-\sqrt{1-n_x^2}, n_x)$
$\alpha = \text{ATAN2}(n_y, -n_z)$
$\gamma = \text{ATAN2}(-O_x, -a_x)$

If $n_x = 1 \rightarrow \beta = 0°$

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha+\gamma) & -\sin(\alpha+\gamma) \\ 0 & \sin(\alpha+\gamma) & \cos(\alpha+\gamma) \end{pmatrix}$$

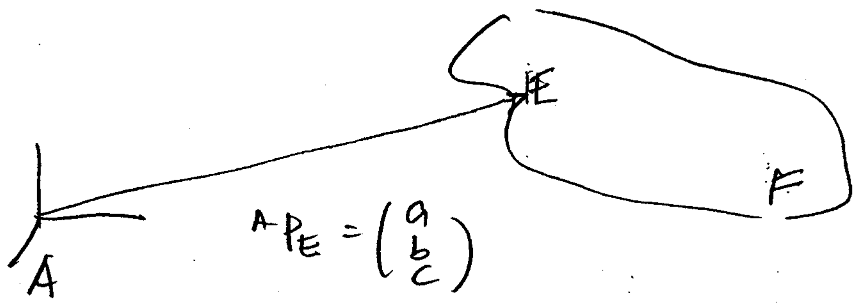
$$\alpha + \gamma = \text{ATAN2}(O_z, O_y)$$ } infinite # of sol'ns for α & γ

If $n_x = -1 \rightarrow \beta = 180°$

$$R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos(\alpha-\gamma) & \sin(\alpha-\gamma) \\ 0 & \sin(\alpha-\gamma) & -\cos(\alpha-\gamma) \end{pmatrix}$$

$$\alpha - \gamma = \text{ATAN2}(O_z, O_y)$$ } infinite # of sol'ns for α & γ

Q2 (b)



$${}^{A}P_E = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$${}^{A}R_E = Rot(x,\alpha)\,Rot(y,\beta)\,Rot(x,\gamma)$$

$${}^{A}\omega_F = {}^{A}\omega_E = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\dot{\alpha} + Rot(x,\alpha)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\dot{\beta} + Rot(x,\alpha)Rot(y,\beta)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\dot{\gamma}$$

$${}^{A}u_F = {}^{A}u_E + {}^{A}\omega_F \times \left({}^{A}R_E\,{}^{E}P_F\right)$$

//

where $${}^{E}T_F = \begin{pmatrix} {}^{E}R_F & {}^{E}P_F \\ 0 & 0 \end{pmatrix}$$

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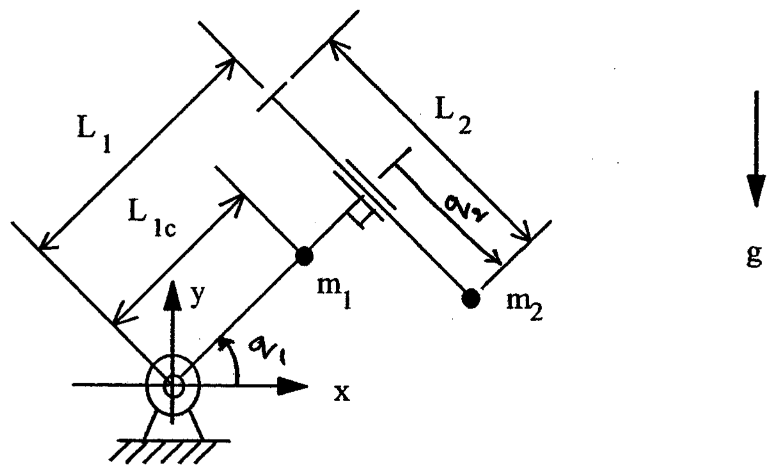
Examination Solutions for:

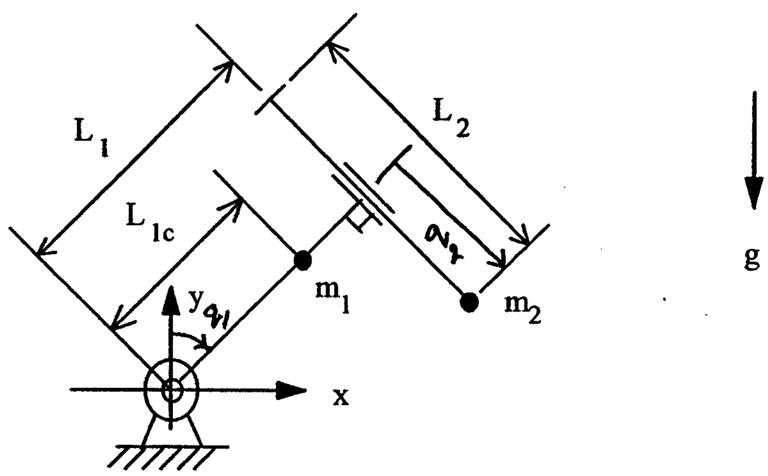
Course No. and Title:	EE4304/ME4245
Semester/Year:	Semester 1, 1999
Solution to Qn No. (& Marks):	3 (25 marks)

Q3 (a)

Set 1.



Set 2.



The first joint is rotational and the second is prismatic.

Q. 3 ⓑ For set 1, the positions of m_1 and m_2 are:

$$m_1 \begin{cases} x_1 = l_{1c} \cos q_1 \\ y_1 = l_{1c} \sin q_1 \end{cases}$$

$$m_2 \begin{cases} x_2 = l_1 \cos q_1 + q_2 \sin q_1 \\ y_2 = l_1 \sin q_1 - q_2 \cos q_1 \end{cases}$$

The velocities of m_1 and m_2 are:

$$\dot{x}_1 = -l_{1c} \sin q_1 \, \dot{q}_1$$

$$\dot{y}_1 = l_{1c} \cos q_1 \, \dot{q}_1$$

$$\dot{x}_2 = -l_1 \sin q_1 \dot{q}_1 + q_2 \cos q_1 \dot{q}_1 + \dot{q}_2 \sin q_1$$

$$\dot{y}_2 = l_1 \cos q_1 \dot{q}_1 + q_2 \sin q_1 \dot{q}_1 - \dot{q}_2 \sin q_1$$

Giving:

$$v_1^2 = \dot{x}_1^2 + \dot{y}_1^2 = l_{1c}^2 \dot{q}_1^2$$

$$v_2^2 = \dot{x}_2^2 + \dot{y}_2^2 = l_1^2 \dot{q}_1^2 + q_2^2 \dot{q}_1^2 + \dot{q}_2^2 - 2 l_1 \dot{q}_1 \dot{q}_2$$

$$= l_1^2 \dot{q}_1^2 + \dot{q}_2^2 - 2 l_1 \dot{q}_1 \dot{q}_2 + q_2^2 \dot{q}_1^2$$

$$= (l_1^2 + q_2^2) \dot{q}_1^2 + \dot{q}_2^2 - 2 l_1 \dot{q}_1 \dot{q}_2$$

$$= (l_1 \dot{q}_1 - \dot{q}_2)^2 + q_2^2 \dot{q}_1^2$$

Q.3 (b) Continued:

The total kinetic energy of the system is:

$$K = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2$$

$$= \tfrac{1}{2} m_1 l_{1c}^2 \dot{q}_1^2 + \tfrac{1}{2} m_2 \left[(l_1^2 + q_2^2)\dot{q}_1^2 + \dot{q}_2^2 - 2 l_1 \dot{q}_1 \dot{q}_2 \right]$$

$$= \tfrac{1}{2} \left\{ \left[m_1 l_{1c}^2 + m_2 (l_1^2 + q_2^2) \right] \dot{q}_1^2 - 2 l_1 \dot{q}_1 \dot{q}_2 + m_2 \dot{q}_2^2 \right\}$$

$$= \tfrac{1}{2} \dot{q}^T D \dot{q}$$

where,

$$D(q) = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix},$$

with,

$$d_{11} = m_1 l_{1c}^2 + m_2 (l_1^2 + q_2^2)$$

$$d_{12} = d_{21} = -m_2 l_1$$

$$d_{22} = m_2$$

The potential energy of the system is:

$$P = l_{1c} m_1 g \sin q_1 + m_2 g (l_1 \sin q_1 - q_2 \cos q_1)$$

The Christoffel symbols defined by,

$$c_{ijk} = \tfrac{1}{2} \left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right],$$

Q. 3 (b) Continued:

are given by:

$$c_{111} = \tfrac{1}{2} \frac{\partial d_{11}}{\partial q_1} = 0$$

$$c_{121} = c_{211} = \frac{1}{2} \frac{\partial d_{11}}{\partial q_2} = m_2 q_2$$

$$c_{221} = \frac{\partial d_{12}}{\partial q_2} - \frac{1}{2} \frac{\partial d_{22}}{\partial q_1} = 0$$

$$c_{112} = \frac{\partial d_{21}}{\partial q_1} - \frac{1}{2} \frac{\partial d_{11}}{\partial q_2} = -m_2 q_2$$

$$c_{122} = c_{212} = \frac{1}{2} \frac{\partial d_{22}}{\partial q_1} = 0$$

$$c_{222} = \frac{1}{2} \frac{\partial d_{22}}{\partial q_2} = 0$$

By definition: $c_{kj} = \sum_{i=1}^{n} c_{ijk} \dot{q}_i$,

giving:

$$c(q, \dot{q}) = \begin{bmatrix} m_2 q_2 \dot{q}_2 & m_2 q_2 \dot{q}_1 \\ -m_2 q_2 \dot{q}_1 & 0 \end{bmatrix}$$

The gravitational force,

$$G(q) = \frac{\partial P}{\partial q} = \begin{bmatrix} m_1 g l_{1c} \cos q_1 + m_2 g (l_1 \cos q_1 + q_2 \sin q_1) \\ -m_2 g \cos q_1 \end{bmatrix}$$

The Complete dynamics is given by,

$$D(q)\ddot{q} + c(q, \dot{q})\dot{q} + G(q) = \tau = [\tau_1 \; \tau_2]^T$$

Q. 3 (c) Computed torque method:

$$\tau = D(q)v + C(q,\dot{q})\dot{q} + G(q) \quad (\quad)$$

where,
v is the new control signal to be defined.

Substituting the above control law into the robot dynamics, a double integrator system is obtained:

$$\ddot{q} = v \quad (\quad)$$

Define $$v_i = -k_{vi}\dot{q}_1 - k_{pi}q_i \quad (\quad)$$

This is a PD control scheme.

Substituting into the double integrator above, the closed loop system is obtained as:

$$\ddot{q}_i + k_{vi}\dot{q}_i + k_{pi}q_i = 0 \; ; \quad i = 1,2$$

Taking Laplace transform:

$$s^2 + k_{vi}s + k_{pi} = 0$$

Comparing with the standard equation:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 .$$

$$\Rightarrow \quad k_{vi} = 2\zeta\omega_n$$

$$k_{pi} = \omega_n^2 = 4^2 = 16$$

For a critically damped system;
$\zeta = 1$.

$\therefore \; k_{vi} = 2 * 1 * 4 = 8.$

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Examination Solutions for:

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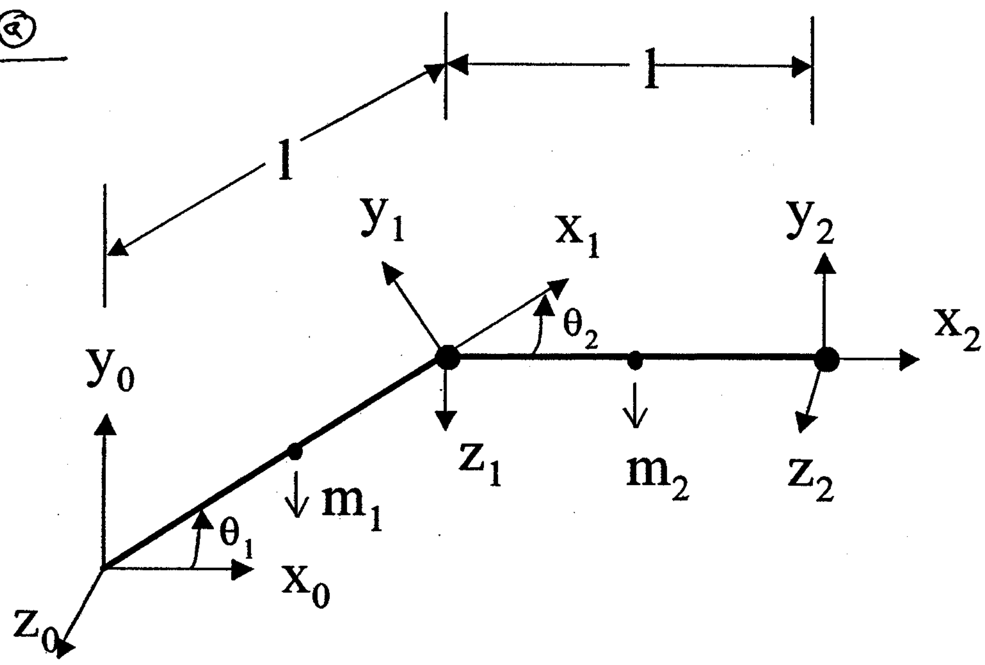
Semester/Year:	Semester 1, 1999
Solution to Qn No. (& Marks):	4 (25 marks)

Q 4.(a)



(i) Using the notation of the L-E equation of motion, we have:

$$D(\theta)\ddot{\theta}(t) + c(\theta,\dot{\theta}) + g(\theta) = \tau(t)$$

where

$$D(\theta) = \begin{bmatrix} d_{11}(\theta_2) & d_{12}(\theta_2) \\ d_{12}(\theta_2) & d_{22} \end{bmatrix},$$

$$c(\theta,\dot{\theta}) = \begin{bmatrix} c_{12}(\theta_2)(\dot{\theta}_2)^2 + 2c_{12}(\theta_2)\dot{\theta}_1\dot{\theta}_2 \\ -c_{12}(\theta_2)\dot{\theta}_1^2 \end{bmatrix} \triangleq \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

and $$g(\theta) = \begin{bmatrix} g_1(\theta_1, \theta_2) \\ g_2(\theta_1, \theta_2) \end{bmatrix}$$

Q.4.(a)(i) Continued.

Since the inverse of $D(\theta)$ always exists (for any practical robot):

$$D^{-1}(\theta) = \frac{1}{d_{11}d_{22}-d_{12}^2}\begin{bmatrix} d_{22} & -d_{12}(\theta_2) \\ -d_{12}(\theta_2) & d_{11}(\theta_2) \end{bmatrix};$$

We have,

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = -D^{-1}(\theta)\left[C(\theta,\dot{\theta}) + g(\theta)\right] + D^{-1}(\theta)\,\tau(t)$$

Choose the state variable vector as

$$x(t) = (\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)^T \triangleq (x_1, x_2, x_3, x_4)^T$$

and the control vector as:

$$u(t) = (\tau_1, \tau_2)^T \triangleq (u_1, u_2)^T$$

Q. 4(a)(ii) From the above equations, the state equations can be obtained:

$$\dot{x}_1(t) = \dot{\theta}_1 = x_3$$

$$\dot{x}_2(t) = \dot{\theta}_2 = x_4$$

$$\dot{x}_3(t) = \ddot{\theta}_1 = \frac{-1}{d_{11}d_{22}-d_{12}^2}\left[d_{22}(c_1+g_1) - d_{12}(c_2+g_2) - u_1\right]$$

$$\dot{x}_4(t) = \ddot{\theta}_2 = \frac{-1}{d_{11}d_{22}-d_{12}^2}\left[-d_{12}(c_1+g_1) + d_{11}(c_2+g_2) - u_2\right]$$

Q.4 .(a) (ii) Continued

Expressed in matrix form, we have,

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

$$- \frac{1}{d_{11}d_{22} - d_{12}^2}\begin{bmatrix} 0 \\ 0 \\ d_{22}(c_{12}\dot{\theta}_2^2 + 2c_{12}\dot{\theta}_1\dot{\theta}_2 + g_1) - d_{12}(-c_{12}\dot{\theta}_1^2 + g_2) \\ -d_{12}(c_{12}\dot{\theta}_2^2 + 2c_{12}\dot{\theta}_1\dot{\theta}_2 + g_1) + d_{11}(-c_{12}\dot{\theta}_1^2 + g_2) \end{bmatrix}$$

$$+ \frac{1}{d_{11}d_{22} - d_{12}^2}\begin{bmatrix} 0 \\ 0 \\ u_1(t) \\ u_2(t) \end{bmatrix}$$

Q.4. (b) The basic idea of independent joint control schemes is:

Assume that each joint is modeled by

$$\bar{m}_i \ddot{q}_i = \tau_i$$

where $\bar{m}_i$ is the nominal inertia value of joint i. Then the well known PID Controller of the form,

$$\tau_i = k_{Di}\dot{q}_i + k_{Pi}q_i + k_{Ii}\int_0^t q_i \, dt$$

can be used.

Q. 4 (b) Continued

The advantages of the control scheme are:

(i) Simple and easy to implement
(ii) robust in terms of parameter variations.

The disadvantages of the scheme are:

(i) The closed loop performance is not very good, as the coupling is not compensated for
(ii) The control performance varies for different loads and different robot configurations.

Q. 4. (c) Advantages: Conceptually simple, easy to design and, it is easy to specify the design performance

Disadvantages: Exact knowledge about the system is required, Computationally intensive, and not robust to unmodeled dynamics.

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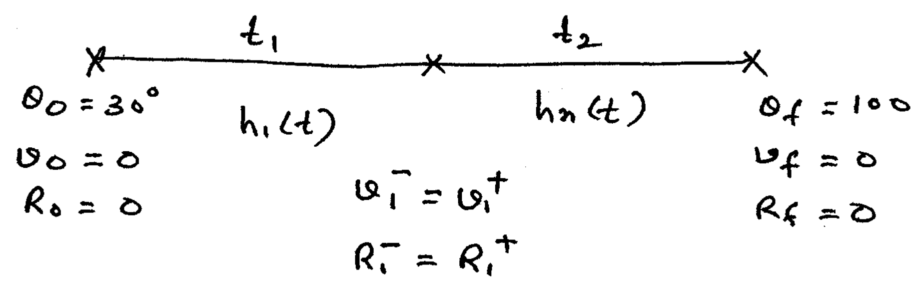
Solution to Qn No. (& Marks):	5 (25 marks)

Q. 5 (a) (i) 5^{th} degree polynomial unless the trajectory is split into several trajectory segments. In this case, 3^{rd} degree polynomial is the lowest

(ii) For a quatric polynomial, we have 5 unknowns, but with 6 constraints. So it is required to split the trajectory into two segments. An intermediate point can be picked and with the constraints the unknowns can be found.



For each trajectory segments (2 segments) we have a quatric polynomial.

Q. 5 (a) ii : Continued

For the first segment:

$$h_1(t) = a_{14}t^4 + a_{13}t^3 + a_{12}t^2 + a_{11}t + a_{10}, \quad t \in [0,1].$$

Using the position, velocity and acceleration constraints at $t=0$:

$$h_1(t) = a_{14}t^4 + a_{13}t^3 + \theta_0,$$

where $\theta_0 = 30^\circ$.

For the second segment:

$$h_n(t) = a_{n4}t^4 + a_{n3}t^3 + a_{n2}t^2 + a_{n1}t + a_{n0},$$
$$t \in [-1,0]$$

Using the position, velocity and acceleration constraints at $t=0$:

$$h_n(t) = a_{n4}t^4 + a_{n3}t^3 + \theta_f$$

where $\theta_f = 100^\circ$.

To have continuity between segments, the velocity and acceleration from the first segment must be continuous with that at the beginning of the second segment:-

$$v_1^- = \frac{\dot{h}_1}{t_1}\bigg|_{t=1} = \frac{4a_{14} + 3a_{13}}{t_1}$$

$$v_1^+ = \frac{\dot{h}_n}{t_n}\bigg|_{t=-1} = \frac{-4a_{n4} + 3a_{n3}}{t_2}$$

Q.5 (a)(ii) Continued

and,

$$a_1^- = \frac{\ddot{h}_1}{t_1^2}\Big|_{t=1} = \frac{12a_{14} + 6a_{13}}{t_1^2}$$

$$a_1^+ = \frac{\ddot{h}_n}{t_n^2}\Big|_{t=-1} = \frac{12a_{n4} - 6a_{n3}}{t_2^2}$$

Equating: $v_1^- = v_1^+$ and $a_1^- = a_1^+$, give two equations with 4 unknowns:

At the selected intermediate position, the difference of joint angles between the two segments are,

$$\delta_1 = h_1(1) - h_1(0) = a_{14} + a_{13}$$
$$\delta_n = h_n(0) - h_n(-1) = -a_{n4} + a_{n3}$$

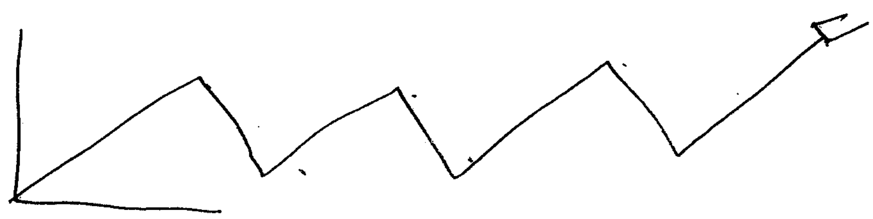
The 4 unknown equations can be expressed as:

$$\begin{bmatrix} 3/t_1 & 4/t_1 & -3/t_2 & 4/t_2 \\ 6/t_1^2 & 12/t_1^2 & 6/t_2^2 & -12/t_2^2 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{14} \\ a_{n3} \\ a_{n4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \delta_1 \\ \delta_n \end{bmatrix}$$

$\Rightarrow$ give the solution.

For the second segment, the normalized time is from $t \in [-1, 0]$ and has to be changed back to $t \in [0, 1]$ by substituting $t = (t-1)$ in the expression for $h_n(t)$.

Q5(b)

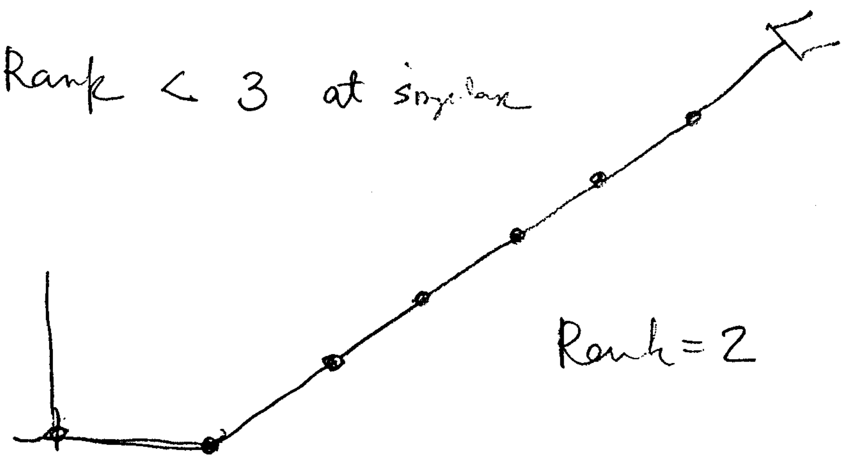


(i) 3 doF - 2 position (x, y)

- 1 orientation about z

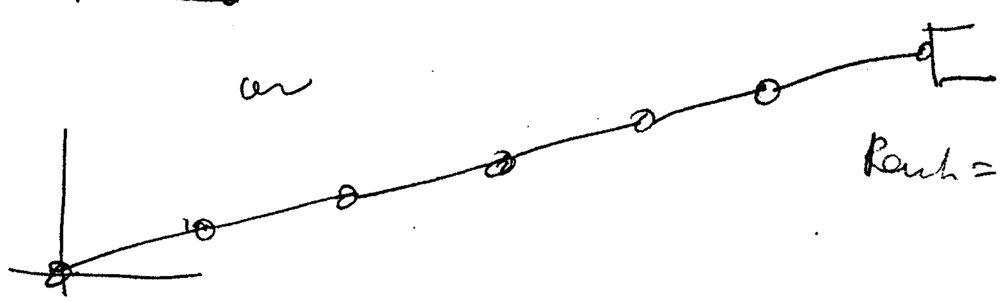
(ii) Rank < 3 at singular

(iii)



Rank = 2

or



Rank = 1

(iv) 3 joints