Linear Systems with Hard Constraints and Variable Set Points:
Their Robustly Invariant Sets

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Abstract

This report considers the properties and computation of $O_\infty(r)$, the maximal robustly invariant set for a linear system with set point $r$, hard constraints and set-bounded disturbances. The objective is a straightforward and self-contained review and elaboration of prior results.

I. INTRODUCTION

This note considers the system

$$x(t + 1) = Ax(t) + Br + E_xw(t),$$
(1)

$$y(t) = Cx(t) + Dr + E_yw(t) \in Y, t \in Z^+, (2)$$

where $x(t) \in \mathbb{R}^n$, $r \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^p$, $Y \subset \mathbb{R}^r$ and $Z^+$ is the set of non-negative integers. Here $r$ is a constant set point, $w(t)$ is a disturbance input, and $y(t) \in Y$ imposes a hard constraint on the evolution of (1) over $t \in Z^+$. The disturbance sequence $\{w(t), t \in Z^+\}$, abbreviated by $\{w(t)\}$, belongs to $\mathcal{W} := \{w(t) \in W : t \in Z^+\}$. Typically, (1) describes a well designed stable feedback system and (2) models rigorous limits on physical variables.

The maximal, constraint-admissible set of initial conditions for (1)-(2) is

$$O_\infty(r) = \{x(0) : (1) \text{ and } (2) \text{ are satisfied for all } t \in Z^+ \text{ and } \{w(t)\} \in \mathcal{W}\}. (3)$$
Clearly, \( x(0) \in O_{\infty}(r) \) requires \( x(1) \in O_{\infty}(r) \). Thus, \( O_{\infty}(r) \) is robustly invariant (RI) : \( x \in O_{\infty}(r) \) implies \( Ax + Br + E_{\xi}w \in O_{\infty}(r) \) for all \( w \in W \). As will be seen, \( O_{\infty}(r) \) is also a robust domain of attraction: there exists a well defined set \( F_{\infty}(r) \), such that \( x(0) \in O_{\infty}(r) \) implies \( x(t) \to F_{\infty}(r) \).

The set \( O_{\infty}(r) \) is well known in the literature \[1\] \[2\] and has a variety of applications, including the synthesis of control schemes for systems with hard constraints \[3\],\[4\],\[5\],\[6\]. The emphasis here is to obtain general results on the properties of \( O_{\infty}(r) \) and its concrete characterization parameterized by \( r \). Similar results appear elsewhere, but, because of their involvement in more general settings, their development is not particularly easy to follow. The presentation here applies under somewhat weaker hypotheses than considered in \[2\] and is mostly self contained.

Mathematical notations used are standard. The identity matrix in \( R^{n \times n} \) is \( I_n \). The superscript \( T \) denotes matrix transpose. The \( i \)th row of \( Q \in R^{n \times n} \) is \( Q_i \in R^{1 \times n} \). For \( x \in R^n \), \( \|x\| \) denotes a suitable norm and \( B_n = \{x : \|x\| \leq 1\} \) is the corresponding unit ball. Set closure and interior are represented respectively by prefixes \textit{cl} and \textit{int}. The set consisting of the single point \( x \) is \( \{x\} \). The inner product of \( x, y \in R^n \) is \( x^T y \). Let \( U, V \subset R^n \) and \( Q \in R^{m \times n} \). Then: \( QU := \{Qx : x \in U\}, U + V := \{x + y : x \in U, y \in V\} \) is the Minkowski sum, \( U \sim V := \{z : z + v \in U, \forall v \in V\} \) is the Minkowski difference \[2\],\[7\]. Since \( U \sim (V_1 + V_2) = (U \sim V_1) \sim V_2 \), there is no ambiguity in writing \( U \sim (V_1 + V_2) = U \sim V_1 \sim V_2 \). If \( U \) is (bounded)[closed]{convex}<polyhedral> then \( QU \) and \( U \sim V \) are (bounded)[closed]{convex}<polyhedral>. If both \( U \) and \( V \) are (bounded)[closed]{convex}<polyhedral> the same properties apply to \( U + V \). Suppose \( U \) is compact and \( \eta \in R^n \). Then \( h_U(\eta) := \max_{x \in U} \eta^T x \) exists and is the support function of \( U \). It follows that \( h_{QU}(\eta) = h_U (Q^T \eta) \) and \( h_{U+V}(\eta) = h_U(\eta) + h_V(\eta) \).

Hereafter, the following assumptions are satisfied: (A1) \( A \) is asymptotically stable; (A2) \((A,C)\) is observable; (A3) \( W \) is compact and \( 0 \in W \); (A4) \( Y \neq \emptyset \) is closed. They can be rationalized in the following ways. (A1): When \( O_{\infty}(r) \neq \emptyset \), system (1),(2) has a robust domain of attraction. (A2): The constraint \( y(t) \subset Y \) does not affect unobservable coordinates of (1)-(2). Thus, it is always possible to obtain a reduced order system where \((A,C)\) is observable. (A3): The compactness of \( W \) is the only simple way of allowing rigorous satisfaction of hard constraints while permitting disturbance variability. The condition \( 0 \in W \) makes results easier to interpret and can always be achieved by a shift in the origin for \( x(t) \). (A4): The comprehensiveness of this assumption allows the treatment of unbounded constraint sets such as those that often occur in applications. Two additional assumptions, of a technical
nature, are introduced in the next section. Assumptions (A5) restricts somewhat the values of $r$; assumption (A2) implies $O_\infty(r)$ is bounded.

The report is organized as follows. Section II introduces notations and definitions crucial to the main results on $O_\infty(r)$. These include definition of the set $F_\infty := F_\infty(0)$ and Theorem 1, which states its main properties. A set, $O_\infty^n \subset R^n \times R^m$, and a recursive procedure for its computation are also introduced. The desired concrete characterization of $O_\infty(r)$ follows directly from $O_\infty^n$. Finally, Section II states the additional two assumptions. Section III is devoted to Theorem 2, which states the main results on $O_\infty^n$ and $O_\infty(r)$. Section IV discusses other sets, related to $O_\infty^n$, that are important in control applications. Section V describes explicit algorithmic procedures for the computation of $O_\infty^n$ when $Y$ is polyhedral.

II. Preliminary Results

The solution of (1) has a useful explicit representation:

$$x(t) = A^t(x(0) - \Gamma r) + \Gamma r + \mu(t). \quad (4)$$

Here $\Gamma r$ is the equilibrium solution of (1) with $w(t) \equiv 0$ and $\mu(t) = \sum_{k=0}^{t-1} A^k E_xW(t-1-k)$, $t \geq 1, \mu(0) = 0$.

Thus,

$$\Gamma := (I - A)^{-1}B \quad (5)$$

and $\mu(t)$ takes on values in the Minkowski sum

$$F_t := \sum_{k=0}^{t-1} A^k E_xW, \quad t \geq 1, \quad F_0 = \{0\}. \quad (6)$$

The sets $F_t$ and $F_\infty$ have important properties.

Theorem 1: Suppose assumptions (A1) and (A3) are satisfied. Then there exists a compact set, $F_\infty$, with the following properties. (i) For all $t \in Z^+$ the sets $F_t$ are compact and satisfy the condition $0 \in F_t \subset F_{t+1} \subset F_\infty$. (ii) Given $\epsilon > 0$, there exists a $t \in Z^+$ such that $F_t \subset F_\infty \subset F_t + \epsilon B_n$. (iii) $F_\infty$ is RI for system (1) with $r = 0$, i.e.,

$x \in F_\infty$ implies $Ax + E_xw \in F_\infty$ for all $w \in W$. (iv) If $X \subset R^n$ is compact and RI for system (1) with $r = 0$, then $F_\infty \subset X$.

The theorem is proved in the Appendix. Simply stated, the main results show $F_t \rightarrow F_\infty$ and $F_\infty$ is minimal over the class of compact RI sets for system (1) with $r = 0$. 
Given any $x(0)$ it follows from (4), $\mu(t) \in F_t$ and $A^t \to 0$ that $x(t)$ converges to the set

$$F_\infty(r) := F_\infty + \{\Gamma r\}. \quad (7)$$

Specifically, given $\epsilon > 0$, there exists a $\tilde{t} \in Z^+$ such that

$$x(t) \in F_\infty(r) + \epsilon B_n, \ \forall t \geq \tilde{t}. \quad (8)$$

Obviously, the properties of $F_\infty$ stated in Theorem 1 apply to $F_\infty(r)$ when $r \neq 0$.

Using (4), (3) becomes

$$O_\infty(r) : = \{x : CA^t(x - \Gamma r) + (CT + D)r + C\mu(t) + E_y w(t) \in Y, \forall t \in Z^+, w(t) \in W, \mu(t) \in F_t\}. \quad (9)$$

The effect of $\mu(t)$ and $w(t)$ in this definition are summarized succinctly by introducing the sets

$$Y_t := Y \sim E_y W \sim CF_t, \ t \in Z^+. \quad (10)$$

Then it is easy to see

$$O_\infty(r) = \{x : CA^t(x - \Gamma r) + (CT + D)r \in Y_t, \forall t \in Z^+\}. \quad (11)$$

By (A4), the sets $Y_t$ and $O_\infty(r)$ are closed. Moreover, $F_t \subset F_{t+1}$ implies $Y_{t+1} \subset Y_t$.

An explicit characterization of $O_\infty(r)$ in the parameter $r$ is obtained from the set

$$O_a^\infty := \{(x, r) : r \in \Omega, CA^t(x - \Gamma r) + (CT + D)r \in Y_t, \forall t \in Z^+\}. \quad (12)$$

In the preceding discussion, there has been no explicit constraint on $r$. That is $\Omega = R^m$. For the present, continue with this assumption. Obviously,

$$O_\infty(r) = \{x : (x, r) \in O_a^\infty\}. \quad (13)$$

Furthermore, $O_a^\infty$ can be represented algorithmically. Let

$$O^a_{k} := \{(x, r) : r \in \Omega, CA^t(x - \Gamma r) + (CT + D)r \in Y_i, i = 0, \ldots, k\}. \quad (14)$$
Then,

\[ O_0^a = \{(x, r) : r \in \Omega, Cx + Dr \in Y_0 \}, \]

\[ O_{k+1}^a = O_k^a \cap \{(x, r) : CA^{k+1}(x - \Gamma r) + (CT + D)r \in Y_{k+1} \}, \] (15)

\[ Y_0 = Y \sim E_y W, \]

\[ Y_{k+1} = Y_k \sim CA^k E_x W; \]

and

\[ O_\infty^a = \bigcap_{k \in \mathbb{Z}^+} O_k^a. \] (16)

Of course, (16) is not suitable for algorithmic computation of \( O_\infty^a \). Two more things are needed: finite termination, the existence of \( \bar{k} \in \mathbb{Z}^+ \) such that \( O_{\infty}^a = O_{\bar{k}}^a \), and an implementable test for determining \( \bar{k} \). Finite determination is a key result of the next section. The test for \( \bar{k} \) is based on an alternative recursion for \( O_k^a \):

\[ O_{k+1}^a := \{(x, r) : (x, r) \in O_k^a, A(x - \Gamma r) + E_x w \in O_k^a \forall w \in W \}. \] (17)

To confirm this recursion, note that by (14) \( O_{k+1}^a = \{(x, r) : r \in \Omega, C(x - \Gamma r) + (CT + D)r \in Y_0, CA^jA(x - \Gamma r) + (CT + D)r \in Y_{j+1}, j = 0, \cdots, k \}. \) Since \( Y_{j+1} = Y_j \sim CA^j E_x W \), this shows that \( O_{k+1}^a = O_0^a \cup \{(x, r) : CA^j(A(x - \Gamma r) + E_x W) + (CT + D)r \in Y_j, j = 0, \cdots, k \} \) which is equivalent to (14).

By (17), \( O_{k+1}^a = O_k^a \) implies \( O_{k+2}^a = O_k^a \) and by repetition \( O_k^a = O_k^a \) for all \( k \geq \bar{k} \). Thus, in practice, recursion (15) continues until \( O_{k+1}^a = O_k^a \) and stops with \( O_\infty^a = O_{\bar{k}}^a \).

It is possible to obtain an explicit characterization for the set

\[ \Omega_d := \{r : O_\infty(r) \neq \emptyset \}. \] (18)

In what follows, suppose \( x(0) \in F_\infty(r) \). Then, by the RI of \( F_\infty(r) \), \( x(t) \in F_\infty(r) \) for all \( t \in \mathbb{Z}^+ \). This in turn implies \( y(t) \in CF_\infty(r) + \{Dr\} + E_y W \) for all \( t \in \mathbb{Z}^+ \). Thus, the set inclusion \( CF_\infty(r) + \{Dr\} + E_y W \subset Y \) implies \( x(0) \in O_\infty(r) \) and \( O_\infty(r) \neq \emptyset \). Now suppose the set inclusion is not satisfied. Then there exist \( x(0) \in F_\infty(r) \) and \( w(0) \in W \) such that \( y(0) = Cx(0) + Dr + E_y w(0) \notin Y \). But this result violates the requirement on \( O_\infty(r) \) that (2) holds for \( t = 0 \). Hence \( O_\infty(r) \neq \emptyset \) if and only if \( r \) satisfies the condition \( CF_\infty(r) + \{Dr\} + E_y W \subset Y \). From
$$\Omega_d := \{r : (CT + D)r \in Y_\infty\}, \quad (19)$$

where

$$Y_\infty := Y \sim E_y W \sim CF_\infty.$$ \quad (20)

It is now possible to state the remaining assumptions:

(A5): The set $\Omega$ is non-empty, compact and satisfies the inclusion

$$(CT + D)\Omega \subset \text{int}Y_\infty. \quad (21)$$

(A6) There exists a $\tilde{t} \in Z^+$ such that $O_0^\infty$ is bounded.

Assumption (A5) implies $\text{int}Y_\infty \neq \emptyset$ and $\Omega \subset \Omega_d$. By (15) and (16), assumption (A6) implies $O_0^\infty$ is bounded. Sometimes the boundedness of $O_0^\infty$ can be verified more directly than by computing $O_t^\infty$ recursively and testing it at each $t$ for boundedness. For example, (A6) is satisfied if $Y$ is bounded, the standard assumption in the literature.

**Remark 1:** To minimize the restriction on $r$ imposed by $\Omega$, $\Omega$ should be a good approximation of $\Omega_d$, the largest set of meaningful set points. By (20) and (A4), $Y_\infty$ is closed, but not necessarily bounded. Thus, $\Omega_d$ is closed but not necessarily bounded. If $\Omega_d$ is unbounded, the compactness of $\Omega$ means a good approximation is not possible. However, in those rare situations where $\Omega_d$ is unbounded, it is usually possible to add fictitious constraints to (2) that make $\Omega_d$ bounded and do not affect materially the utility of problem statement. If $\Omega_d$ is bounded, $\text{int}\Omega_d \neq \emptyset$ and $\text{cl}(\text{int}\Omega_d) = \Omega_d$, a relatively common situation, then $\Omega$ can be an arbitrarily good approximation of $\Omega_d$ that satisfies (21).

### III. The Main Result

**Theorem 2:** Suppose assumptions (A1)-(A6) are satisfied. Then: (i) $O_0^\infty$ is nonempty and compact. (ii) $O_\infty(r)$ is nonempty and compact for all $r \in \Omega$. (iii) $O_0^\infty$ is finitely determined. (iv) There exists $\epsilon > 0$ such that $F_\infty(r) + \epsilon B_n \subset O_\infty(r)$ for all $r \in \Omega$.

**Proof:** (i) and (ii): Assumption (A5) implies $\Omega \neq \emptyset$. Since $\Omega \subset \Omega_d$, $O_\infty(r) \neq \emptyset$ for all $r \in \Omega$. Thus, (13) implies $O_0^\infty$ is non-empty. Since $Y_t$ is closed, representation (12) is an intersection of closed sets. Thus, $O_0^\infty$ is
closed. By the paragraph preceding Remark 1, $O^a_{\infty}$ is bounded. Clearly, representation (11) is an intersection of of closed sets. Thus, $O_{\infty}(r)$ is closed for each $r \in \Omega$. Because $O^a_{\infty}$ is bounded, (13) shows $O_{\infty}(r)$ is bounded.

Proof of (iii): Since $(CT + D)\Omega$ is compact and $Y_{\infty}$ is closed, (A5) implies the existence of $\epsilon' > 0$ such that $(CT + D)\Omega + \epsilon' B_p \subset Y_{\infty}$. Choose $\bar{\epsilon} \in Z^+$ so that $CA^l(x - \Gamma r) \in \epsilon' B_p$ for all $(x, r) \in O^a_{\infty}$ and $t > \bar{\epsilon}$. This choice is possible because $O^a_{\infty}$ is bounded and $A^l \rightarrow 0$. The choice of $\epsilon'$ implies $CA^l(x - \Gamma r) + (CT + D)r \in Y_{\infty} \subset Y_{\bar{\epsilon}}$ for all $t \geq \bar{\epsilon}$ and all $(x, r) \in O^a_{\infty}$. Hence by (12) and (14), $O^a_{\infty} = O^a_{\bar{\epsilon}}$ and $O^a_{\infty}$ is finitely determined.

Proof of (iv): Let $\epsilon'$ be determined as in the previous paragraph. Choose $\epsilon > 0$ so that $\epsilon CA^l B_n \subset \epsilon' B_p$ for all $t \in Z^+$. Result (iv) holds if $r \in \Omega$ and $x(0) \in F_{\infty}(r) + \epsilon B_n$ imply $y(t) \in Y$ for all $t \in Z^+$ and all $\{w(t)\} \in W$. Let $x(0) \in F_{\infty}(r) + \epsilon B_n$ be written as $x(0) = \Gamma r + z(0) + \delta x(0)$ where $z(0) \in F_{\infty}$ and $\delta x(0) \in \epsilon B_n$. Then it is easy to see by the linearity of (1), (2) and the RI of $F_{\infty}$ that $y(t) = CA^l \delta x(0) + (CT + D)r + Cz(t) + \epsilon y w(t)$ where $CA^l \delta x(0) \in \epsilon' B_p$, $(CT + D)r \in (CT + D)\Omega$, $z(t) \in F_{\infty}$ and $w(t) \in W$. Thus, result (iv) is implied by the inclusion $\epsilon' B_p + (CT + D)\Omega + CF_{\infty} + E_y W \subset Y$, which by (20) is equivalent to $\epsilon' B_p + (CT + D)\Omega \subset Y_{\infty}$. □

Remark 2: Theorem 2 and the results of Section II specialize easily to the disturbance-free case [8] where $w(t) \equiv 0$. It is only necessary to set $Y_{\infty} = Y_0 = Y$ and $F_{\infty}(r) = \{\Gamma r\}$.

A few comments on the relationship of Theorem 2 to prior results are in order.

Suppose the assumptions on $Y$ and $\Omega$ are strengthened by adding the requirement that they are (convex)[polyhedral]. Then by (15) and the properties of the Minkowski difference noted in Section 1, $O^a_{\infty}$ and $O_{\infty}(r)$ are (convex)[polyhedral].

The results of Theorem 2 can be obtained from the results in Section 7 of [2]. However, there are significant differences. Details of the proofs are more complex because they allow $r$ to be $t$ dependent where $r(t+1) = A_L r(t)$ and $A_L$ has all its characteristic roots on the unit circle. Assumptions on $Y$ are stronger than here: $0 \in Y$ and $Y$ is compact. The characterization of $\Omega$ differs slightly from (A5): $\Omega := \{r : (CT + D)r \in Y'\}$ where $Y' \subset int Y_{\infty}$ is compact. Further, the interpretation of $\Omega$ as an approximation of $\Omega_d$ is not made clear.

Results similar to the preceding theorem are implicit in papers on command governors. See for example [3], [4]. However, full proofs of main results are lacking and the properties of $O_{\infty}(r)$ and its concrete parametrization in $r$ do not received direct attention. For instance, result (iv) of Theorem 1 is not stated explicitly. The assumptions in
the papers are similar to those in [2], including the compactness of $Y$.

IV. OTHER SETS RELATED TO $O_a^\infty$

The sets $O_\infty(r)$ and $\Omega$ are respectively, the $r$-section of $O_a^\infty$ as determined by (13), and the projection of $O_a^\infty \subset R^n \times R^m$ onto $R^m$:

$$\Omega = \{ r : \exists x \text{ such that } (x, r) \in O_a^\infty \} := \text{Proj}_r O_a^\infty.$$  \hfill (22)

Two other sets, similarly defined in terms of $O_a^\infty$, are also of interest. They are the $x$-section of $O_a^\infty$,

$$\Pi(x) := \{ r : (x, r) \in O_a^\infty \},$$  \hfill (23)

and the projection of $O_a^\infty \subset R^n \times R^m$ onto $R^n$,

$$P_\infty := \{ x : \exists r \text{ such that } (x, r) \in O_a^\infty \} := \text{Proj}_x O_a^\infty.$$  \hfill (24)

Figure 1 gives a schematic representation for all of the sets.

![Schematic representation of sets](image)

Fig. 1. A schematic representation of the sets $O_a^\infty$, $\Omega$, $O_\infty(r)$, $P_\infty$ and $\Pi(r)$.

Clearly,

$$P_\infty = \bigcup_{r \in \Omega} O_\infty(r).$$  \hfill (25)

Thus, $P_\infty$ is the maximal set of initial states that can be captured by $O_\infty(r)$ for set points $r \in \Omega$. Unlike $\Omega$, the set $P_\infty$ does not have a simple concrete characterization. There are difficulties even when $O_a^\infty$ is a polyhedron,
as described in the next section. Algorithms then exist that in principle implement the projection operation and construct a set of linear inequalities that defines the polyhedron $P_\infty$. Unfortunately, unless $n$ is small, the related computations are generally expensive and numerically troublesome.

Obviously, $\Pi(x) \subset \Omega$ and is nonempty if and only if $x \in P_\infty$. Since $\Pi(x)$ consists of all those $r$ such that $x \in \Omega \cap (r)$, it has important implications in the synthesis of control laws for system (1) where $r$ is replaced by $u(t) = U(x(t))$. Suppose the control law $U(x)$ satisfies the condition: $U(x) \in \Pi(x)$ for all $x \in P_\infty$. Then, $x \in P_\infty$ implies $x \in \Omega \cap (U(x))$ and the RI of $\Omega \cap (U(x))$ implies $Ax + BU(x) + E_\infty w \in \Omega \cap (U(x))$ for all $w \in W$. Thus, $P_\infty$ is a RI set for the closed loop system, (1) with $r = U(x)$. It is also a constraint-admissible set for the closed-loop system: $x(0) \in P_\infty$ implies $y(t) \in Y$ for all $t \in Z^+$ and $\{w(t)\} \in W$. Of course, the condition $U(x) \in \Pi(x)$ allows great freedom in the choice of $U(x)$. Additional desirable results, such as convergence of $x(t)$ to $F_\infty$, require additional conditions on $U(x)$.

Note $x \in P_\infty$ if and only if $\Pi(x) \neq \emptyset$. The equivalence offers an alternative when $x \in P_\infty$ must be tested and an explicit representation for $P_\infty$ is not available. The equivalent testing of $\Pi(x) \neq \emptyset$ is often easy because $\Pi(x) \subset R^m$ and $m$ is typically small.

V. Computational Issues

When $Y$ is polyhedral, recursions (15) are well suited to practical computations. Given $\mu \in Z^+, S \in R^{\mu \times p}, s \in R^\mu$ and

$$Y = \{y : Sy \leq s\} = \{y : S_i y \leq s_i, i = 1, \cdots, \mu\},$$

(26)
algorithms exist for determining $\nu \in Z^+, H_x \in R^{\nu \times n}, H_r \in R^{\nu \times n}$ and $h \in R^\nu$ such that

$$O_\infty^\nu = \{(x, r) : H_x x + H_r r \leq h\}.$$  

(27)

Thus,

$$O_\infty(r) = \{x : H_x x \leq h - H_r r\}$$

(28)

and

$$\Pi(x) = \{r : H_r r \leq h - H_x x\}.$$  

(29)
This section reviews details of the procedures critical to their implementation as computer routines.

The first requirement is an explicit algorithmic representation for the polyhedron $Y_k$. It exploits an easily proved identity [2], [7] for Minkowski subtraction from the polyhedron $Y$: given a compact set $Z \in \mathbb{R}^p$, $S \sim Z = \{y : S_i y \leq s_i - h_Z(S_i^T), i = 1, \ldots, \mu\}$. By this identity, (10), and properties of support functions stated in Section I, $Y_k = \{y : S_i y \leq s_i - h_w(E_y^T S_i^T) - h_{f_k}(C_i^T S_i^T), i = 1, \ldots, \mu\}$. Thus, $Y_k$ can be written

$$Y_k = \{y : S_i y \leq s^{k_i}\} \begin{equation} (30) \end{equation}$$

Using this form of $Y_k$ in recursion (15) yields a recursion for the $s^{k_i}$:

$$s_0^{k_i} = s_i - h_w(E_y^T S_i^T), s^{k_i+1}_i = s^{k_i} - h_w((CA^k E_x)S_i^T). \begin{equation} (31) \end{equation}$$

The determination of $Y_k$ by (30) and (31) circumvents the need to compute $F_k$. This is important because the Minkowski sums in (6) have high operation counts and are sensitive to rounding errors. Since $0 \in W$, the values of the support functions in (31) are non-negative. Thus, the sequences $\{s^{k_i} : i \in \mathbb{Z}^+\}$ are non-increasing. The set $Y_\infty$ is needed, in assumption (A5), to determine the set $\Omega$. It is given by

$$Y_\infty = \{y : S y \leq s^{\infty}\}, \begin{equation} (32) \end{equation}$$

where $s^{\infty}_i = \lim s^{k_i}$ as $k \to \infty$. The limits $s^{\infty}_i$ exist because the sequences $\{s^{k_i}\}$ are non-increasing. In fact, the sequences are exponentially convergent because $h_w((CA^k E_x)S_i^T)$ converges exponentially to 0. Thus, it is feasible to determine $s^{\infty}$ to a high accuracy by using (31) and choosing $k$ reasonably large. Moreover, if $\text{int}Y_\infty$ is nonempty, $Y_\infty$ is bounded, and $\epsilon > 0$ is sufficiently small, a suitable choice for $\Omega$ is

$$\Omega = \{r : S(C \Gamma + D) \leq s^{\infty}(\epsilon)\}, s^{\infty}_i(\epsilon) := s^{\infty}_i - \epsilon, \; i = 1, \ldots, \mu. \begin{equation} (33) \end{equation}$$

Thus, as $\epsilon \to 0$, $\Omega$ becomes an arbitrarily good approximation of

$$\Omega_d = \{r : S(C \Gamma + D) \leq s^{\infty}\}. \begin{equation} (34) \end{equation}$$

With the preceding polyhedral characterizations of $Y_k$ and $\Omega$ the recursions for $O_{a_k}^a$ in (15) can be implemented. For compactness of notation it is convenient to write $O_{k+1}^a = O_k^a \cap \Phi_{k+1}$, where,

$$\Phi_k := \{(x, r) : CA^k(x - \Gamma r) + (C \Gamma + D)r \in Y_k\}$$

$$= \{(x, r) : G_a^k x + G_r^k \leq s^k\}. \begin{equation} (35) \end{equation}$$
and

\[ G^k_x := SCA^k, \quad G^k_r := S(CT + D) - G^k_x \Gamma. \]

(36)

Finally, for \( k \in \mathbb{Z}^+ \), let

\[ O^0_a = \{(x, r) : H^0_x x + H^0_r x \leq h^0 \}. \]

(37)

Then, it follows from (33) and (15) that

\[
\begin{bmatrix}
SC \\
0
\end{bmatrix},
\begin{bmatrix}
SD \\
S(CT + D)
\end{bmatrix},
\begin{bmatrix}
s^0 \\
s^\infty(\epsilon)
\end{bmatrix}
\]

(38)

and

\[
\begin{bmatrix}
H^k_x \\
G^k_x + 1
\end{bmatrix},
\begin{bmatrix}
H^k_r \\
G^k_r + 1
\end{bmatrix},
\begin{bmatrix}
h^k \\
s^{k+1}
\end{bmatrix}
\]

(39)

Recursion (38)-(39) continues until there exists a \( \tilde{k} \) such that \( O^a_{\tilde{k}+1} = O^a_{\tilde{k}} \). Then, \( H_x = H^k_x, H_r = H^k_r \) and \( h = h^k \), terminating the algorithmic process.

Typically, many of the \( \mu(k+2) \) rows in (37) correspond to redundant inequalities in the characterization of \( O^a_k \).

Rather than letting the number of rows increase unnecessarily as \( k \) increases, it is better to eliminate redundant rows from the inequalities \( H^k_x x + H^k_r r \leq h^k \) each time \( k \) is increased. This is done in (39), one row at a time, starting from the top row of the inequalities defined by \( G^k_x x + G^k_r r \leq s^k \). Each row is tested for redundancy by solving a linear programming on the polyhedron formed from all rows in \( H^k_x x + H^k_r r \leq h^k \) preceding the row being tested. Rows proving to be redundant are discarded. Clearly, this row elimination process does not upset the validity of the overall algorithmic process and, when termination occurs at \( \tilde{k} \), there are no redundant rows left in the representation of \( O^a_{\tilde{k}} \). The elimination of redundant rows at each increase in \( k \) is generally much faster and numerically more stable than waiting to eliminate all redundant rows at \( \tilde{k} \).

For \( \Omega \equiv \Omega_d, \epsilon > 0 \) should be small. In practical computations, the complexity of representation (27) as determined by the number of non-redundant rows, \( \nu \), depends on \( \epsilon \). As \( \epsilon \) increases, \( \nu \) decreases. Thus, there is a tradeoff between size of \( \Omega \) and the complexity of (10).

An alternative setup for the computation of \( O^a_\infty \) permits direct application of available software [9],[10]. Replace
the set point $r$ in (1) by $v(t)$, where $v(t+1) = v(t)$ and $v(0) = r$. This yields the system

$$z(t+1) = A^a z(t) + E_x^a w(t), \quad y(t) = C^a z(t) + E_y w(t),$$

(40)

where

$$A^a := \begin{bmatrix} A & B \\ 0 & I_m \end{bmatrix}, \quad E_x^a := \begin{bmatrix} E_x \\ 0 \end{bmatrix}, \quad C^a := \begin{bmatrix} C & D \end{bmatrix}, \quad z(t) = \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}.$$  

(41)

The constraints $y(t) \in Y$ and $r \in \Omega$ define an equivalent constraint set for $z(t)$:

$$Z^a := \{z : SC^a z \leq s^0, S(C \Gamma + D) \begin{bmatrix} 0 & 0 \\ 0 & I_m \end{bmatrix} z \leq s^\infty(\epsilon)\}.$$  

(42)

Thus, $O^a_\infty$ is defined by a system of the form (40) with a polyhedral constraint on state:

$$O^a_\infty = \{(x, r) : z(0) = \begin{bmatrix} x \\ r \end{bmatrix}, z(t) \in Z^a \text{ for all } t \in Z^+ \text{ and } \{w(t)\} \in W\}.$$  

(43)

This is precisely the situation required by the aforementioned software. Two comments apply. To guarantee finite determination the software requires all characteristic roots of $A^a$ to have magnitude less than 1. While this requirement is not satisfied by our $A^a$, it follows from assumption (A5) and Theorem 1 that $O^a_\infty$ is finitely determined. Of course, $\epsilon > 0$ must be chosen to assure (A5). Since the software does not exploit the special structure of $A^a, E_x^a$ and $C^a$, the computations are not as efficient as those based on (38) and (39).

By Remark 2 the deterministic case leads to minor changes in the main algorithmic steps in the preceding procedures. Specifically, $s^k = s$ for all $k \in Z^+$ and $s_i^\infty(\epsilon) = s_i - \epsilon$ for $i = 1, \ldots, \mu$.

REFERENCES


Proof of Theorem 1. Since the sets $F_t, t \in Z^+$, are compact they belong, with the Hausdorff set metric $\rho$, to a complete metric space [Aubin 97]. From (6), $F_{t+1} = F_t + A^t E_x W$. Therefore, $\rho(F_t, F_{t+1}) \leq \rho(\{0\}, A^t E_x W)$ and by the compactness of $W$ and $A^t \to 0$, $\rho(F_t, F_{t+1}) \to 0$. Thus, $\{F_t : t \in Z^+\}$ is Cauchy and the existence of the limit, $F_\infty$, is established. The first two inclusions of (i) are obvious from (6) and the properties of Minkowski summation. Suppose, contrary to the last inclusion of (i), there exist $\bar{t}$ and $\bar{x} \in F_{\bar{t}}$ such that $\bar{x} \notin F_\infty$. Then, $\bar{x} \in F_t$ for all $t \geq \bar{t}$. Thus by (6) and $F_t \to F_\infty$, $\bar{x} \in F_\infty$ and the contradiction completes the proof of (i). Result (ii), is restatement of $\rho(F_t, F_\infty) \to 0$. From $F_{t+1} = AF_t + E_x W$ and $\rho(F_t, F_\infty) \to 0$, it follows that $F_\infty = AF_\infty + E_x W$. This equality implies result (iii). Assume, contrary to (iv), there exists $z \in F_\infty$ such that $x \notin X$. Then because $X$ is compact there exists $\epsilon > 0$ such that $(\{z\} + \epsilon B_n) \cap X = \emptyset$. Let $x(0) \in X$. Because $z \in F_\infty$ and $A^t \to 0$ there exists a $\{w(t)\} \in \mathcal{W}$ such that $x(t) \to z$. Hence, $x(t)$ eventually leaves $X$ and the RI of $X$ is violated.