We observe that the first three joint axes always intersect at the common point $O_0 = O_1 = O_2$. A closed-form inverse kinematic solution is therefore guaranteed. We use the decoupling principle to arrive at the solution. Since the location of the co-intersection point $O_0 = O_1 = O_2$ as seen from the end-effector frame of reference (frame 6) does not change with motion of the first three joints, the position of $O_0 = O_1 = O_2$ expressed in the end-effector frame (frame 6), $^6p_0 = ^6p_1 = ^6p_2$, is a function of the last three joint coordinates (joints 4, 5, and 6) only. This position $^6p_0$ is directly obtained from the task description $^0T_6$ using:

$$^6p_0 = -^0R_6^T^0p_6,$$  

(1)

i.e., $^6p_0$ is the last column of the inverse of $^0T_6$. We express $^6p_0$ in terms of the last three joint coordinates by taking the fourth column of $^0T_2 = A_6^{-1}A_5^{-1}A_4^{-1}$:

$$^6p_0 = ^6p_2 = \begin{pmatrix} p_x^6 \\ p_y^6 \\ p_z^6 \end{pmatrix} = \begin{pmatrix} -c_6(L_1c_45 + L_2c_5) \\ s_6(L_1c_45 + L_2c_5) \\ L_1s_45 + L_2s_5 - L_3 \end{pmatrix}$$  

(2)

The decoupled system consists of the decoupled task (Equation (2)) and decoupled set of joint coordinates 4, 5 and 6 (Equation (1)).

The nonlinear system (2) represents a low-order system of three equations in three unknowns $(\theta_4, \theta_5, \theta_6)$, for which a closed-form solution is guaranteed:

$$\theta_4 = ATAN2 \left( \frac{\sqrt{4L_1^2L_2^2 - (p_x^6)^2 + (p_y^6)^2 + (p_z^6)^2 - L_1^2 - L_2^2}}{p_x^6 + (p_y^6)^2 + (p_z^6)^2 - L_1^2 - L_2^2} \right)$$  

(3)

where $4L_1^2L_2^2 - (p_x^6)^2 + (p_y^6)^2 + (p_z^6)^2 - L_1^2 - L_2^2$ must be $\geq 0$; otherwise, the end-effector position is unreachable.

$$\theta_5 = ATAN2 \left( \frac{I_6p_y^6}{-I_6p_x^6} \right)$$  

(4)

$$\theta_6 = ATAN2 \left( \frac{L_1c_4 + L_2c_5 - L_1s_4(-p_x^6c_6 + p_y^6s_6)}{(L_1c_4 + L_2c_5)(-p_x^6c_6 + p_y^6s_6) + L_1s_4(p_x^6 + L_3)} \right)$$  

(5)

where $I_4 = \pm 1$ in (3) and $I_6 = \pm 1$ in (4). Having solved for $(\theta_4, \theta_5, \theta_6)$, we now compute

$$^3T_6 = A_4A_5A_6.$$  

(6)

The orientation of frame 3 in frame 0 is then computed from the the task description:

$$^0T_3 = ^0T_6^3T_6^{-1}.$$  

(7)
\( 0^0T_3 \) is a function of the first three joint coordinates only. Having solved for \((\theta_4, \theta_5, \theta_6)\) to satisfy the end-effector position, we know solve for \((\theta_1, \theta_2, \theta_3)\) to satisfy the end-effector orientation by taking the rotation matrix part of \(0^0T_3 = A_1A_2A_3\) only:

\[
0^0R_3 = \begin{pmatrix} N_x & O_x & A_x \\ N_y & O_y & A_y \\ N_z & O_z & A_z \end{pmatrix} = \begin{pmatrix} c_1c_2c_3 + s_1s_3 & -c_1c_2s_3 + s_1c_3 & -c_1s_2 \\ s_1c_2c_3 - c_1s_3 & -s_1c_2s_3 - c_1c_3 & -s_1s_2 \\ -s_2c_3 & s_2s_3 & -c_2 \end{pmatrix} \quad (8)
\]

From the \(A_z\) elements we have

\[
\theta_2 = ATAN2 \left( \frac{I_2\sqrt{1 - A_z^2}}{-A_z} \right), \quad (9)
\]

where \(I_2 = \pm 1\). From the \(A_x, A_y, N_z,\) and \(O_z\) elements of (8), and for \(A_z^2 \neq 1\), we have:

\[
\theta_1 = ATAN2 \left( \frac{-I_2A_y}{-I_2A_x} \right), \quad (10)
\]

\[
\theta_3 = ATAN2 \left( \frac{I_2O_z}{-I_2N_z} \right). \quad (11)
\]

In general, there are eight inverse kinematic solutions corresponding to \(I_4 = \pm 1\), \(I_6 = \pm 1\), and \(I_2 = \pm 1\).

If \(A_z = \pm 1\), \(\theta_1\) and \(\theta_3\) describe the same rotation and cannot be computed separately; one degree-of-freedom is lost.

\[
A_z = -1 \leftrightarrow \begin{cases} 
\theta_2 = 0^0 \\
\theta_1 - \theta_3 = ATAN2 \left( \frac{O_x}{N_x} \right)
\end{cases} \quad (12)
\]

\[
A_z = 1 \leftrightarrow \begin{cases} 
\theta_2 = 180^0 \\
\theta_1 + \theta_3 = ATAN2 \left( \frac{O_x}{-N_x} \right)
\end{cases} \quad (13)
\]

Thus when \(A_z = \pm 1\), there are an infinite number of solutions.