CHAPTER 1

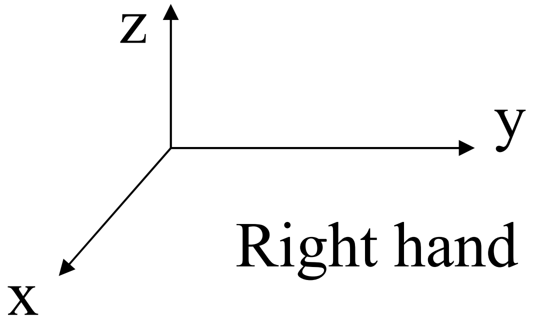
Position and Orientation Definitions and Transformations

Learning Objectives

- Describe position and orientation of rigid bodies relative to each other, and to some reference
- Mathematically represent position and orientation of rigid bodies
- Be able to physically visualize these mathematical representations

SPATIAL DESCRIPTION

- used to specify spatial attributes of various objects with which a manipulation system deals
- universe coordinate frame is implicit
  - Reference (with respect to where?)
- use Cartesian coordinate frames
  - (x, y and z axes = we will use unit vectors i, j, k, respectively)

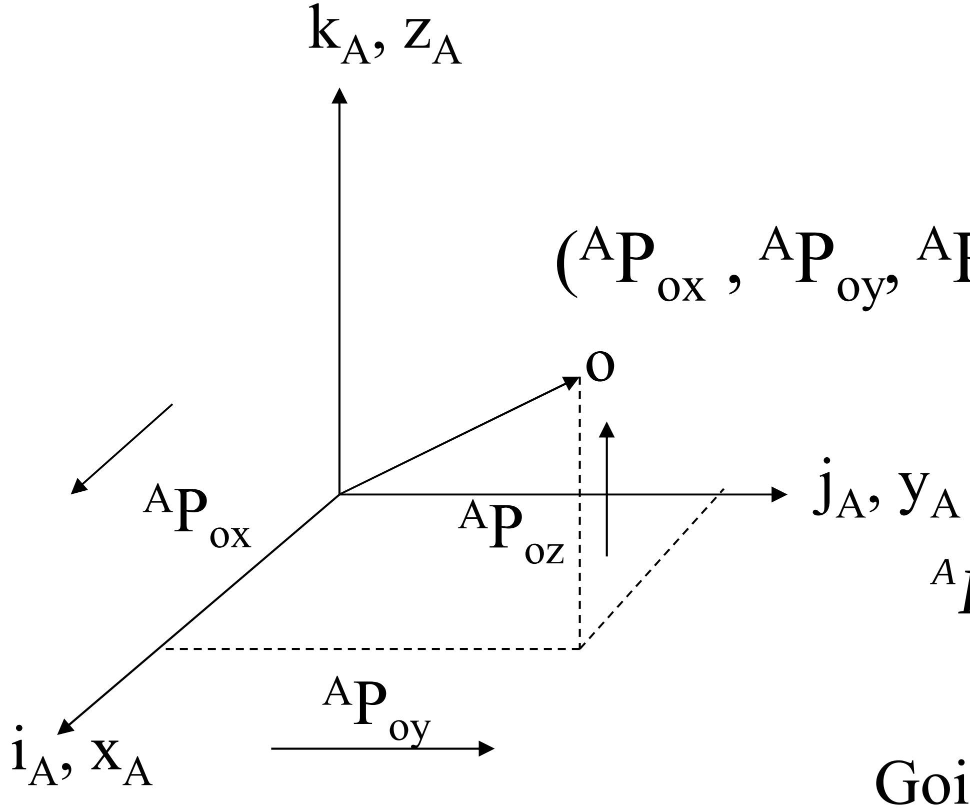


Right hand rule

POSITION

- Attribute of a point
- Need a reference frame

$$ {}^{A}P_{o} = \begin{bmatrix} {}^{A}P_{ox} \\ {}^{A}P_{oy} \\ {}^{A}P_{oz} \end{bmatrix} \in \Re^{3} $$



$({}^{A}P_{ox}, {}^{A}P_{oy}, {}^{A}P_{oz})$ = Cartesian coordinates of pt.0 expressed in frame A

$$ {}^{A}P_{o} = {}^{A}P_{o,x}\ {}^{A}\vec{i} + P_{o,y}\ {}^{A}\vec{j} + P_{o,z}\ {}^{A}\vec{k} $$

Vector addition

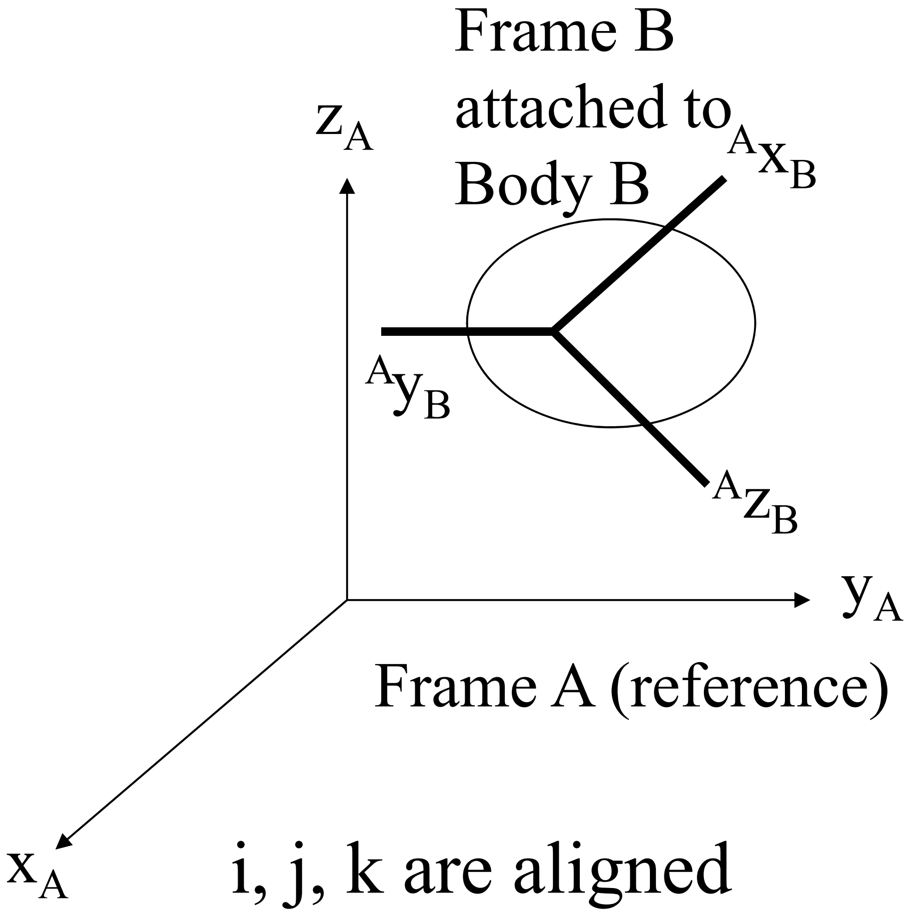
Going from A to O.

Position vector - (coords has magnitude & direction, origin impt)

Orientation of a Body

- attribute of a body, relative to some reference frame
- Need a "reference"
  - attach a Cartesian coordinate frame to the body (i,j,k axes)
- Mathematical description: orientation of each axis of the Cartesian coordinate frame
  - free vectors = orientation vectors = describe orientation
    - magnitude & origin (where it originates from and ends) not important
    - only direction is important

ORIENTATION



$({}^{A}i_{B,}\ {}^{A}j_{B,}\ {}^{A}k_{B})$

– unit vectors along each axis of frame B

– free vectors, only direction is relevant, expressed in A

$$
{}^{A}R_{B} = \left({}^{A}i_{B} \;\; {}^{A}j_{B} \;\; {}^{A}k_{B} \right)
= \begin{pmatrix} {}^{A}i_{Bx} & {}^{A}j_{Bx} & {}^{A}k_{Bx} \\ {}^{A}i_{By} & {}^{A}j_{By} & {}^{A}k_{By} \\ {}^{A}i_{Bz} & {}^{A}j_{Bz} & {}^{A}k_{Bz} \end{pmatrix} \in \Re^{3\times 3}
$$

i, j, k are aligned with the x, y and z axes

Rotation Matrix

- Since each column of the rotation matrix represents unit vectors along the x, y and z directions of the Cartesian coordinates, they should be orthogonal to each other
- Each column obeys unit length constraints
- Cartesian coord frame : right hand rule

$$\therefore \det({}^{A}R_{B}) = +1$$

${}^{A}R_{B}$ is a Proper Orthogonal Matrix

${}^{A}R_{B}^{-1} = {}^{A}R_{B}^{T} = {}^{B}R_{A}$ or ${}^{A}R_{B}\,{}^{A}R_{B}^{T} = I$

Special Case: Inverse of matrix = Transpose of matrix

Rotation Matrix

- All possible orientations of a rigid body (i.e. coordinate frame attached to the body) can be uniquely specified by a rotation matrix
  - Physical visualization -> unique 3x3 matrix
  - 3x3 matrix -> unique physical visualization

- Orientation has 3 degrees-of-freedom in 3D space, i.e., 3 independent parameters are only needed

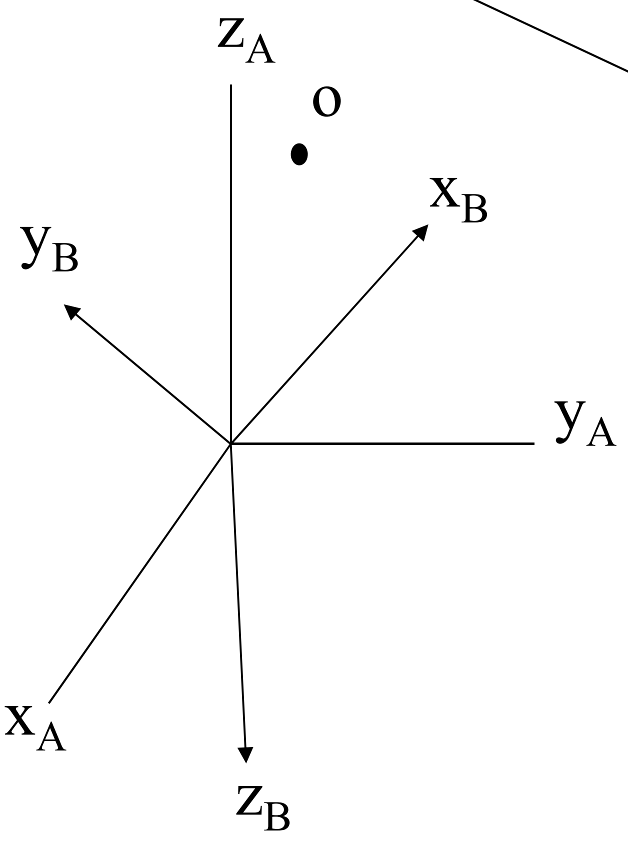
Examples – Position and Orientation

Learning Objectives

- (1) Transformation of position and orientation:
  - Coordinates Expressed in Frame A → Transformed to be expressed in Frame B
- (2) New position and orientation after motion (i.e., rigid body motion)
- Duality between **transformations** and **rigid body motions (1 and 2)**

Coordinates of pt O in A and B

What is this matrix?



Position of O in A

$${}^AP_o = {}^AP_{o,x}\;{}^A\overrightarrow{i_A} + P_{o,y}\;{}^A\overrightarrow{j_A} + P_{o,z}\;{}^A\overrightarrow{k_A} =$$

$$\begin{pmatrix} {}^Ai_{A,x} & {}^Aj_{A,x} & {}^Ak_{A,x} \\ {}^Ai_{A.y} & {}^Aj_{A,y} & {}^Ak_{A,y} \\ {}^Ai_{A,z} & {}^Aj_{A,z} & {}^Ak_{A,z} \end{pmatrix} \begin{pmatrix} {}^AP_{o,x} \\ {}^AP_{o,y} \\ {}^AP_{o,z} \end{pmatrix}$$

Position of O in B

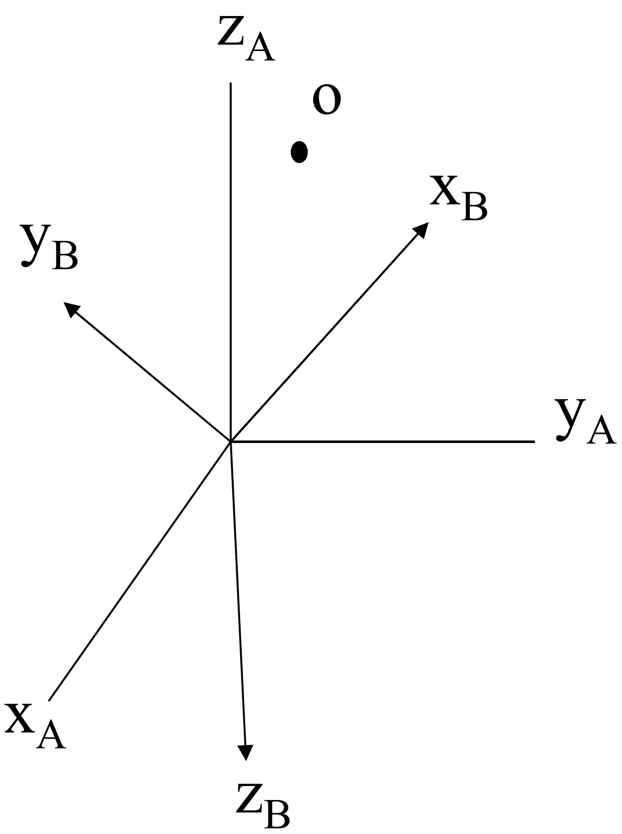
$${}^BP_o = {}^AP_{o,x}\;{}^B\overrightarrow{i_A} + {}^AP_{o,y}\;{}^B\overrightarrow{j_A} + {}^AP_{o,z}\;{}^B\overrightarrow{k_A} =$$

$$\begin{pmatrix} {}^Bi_{A,x} & {}^Bj_{A,x} & {}^Bk_{A,x} \\ {}^Bi_{A,y} & {}^Bj_{A,y} & {}^Bk_{A,y} \\ {}^Bi_{A,z} & {}^Bj_{A,z} & {}^Bk_{A,z} \end{pmatrix} \begin{pmatrix} {}^AP_{o,x} \\ {}^AP_{o,y} \\ {}^AP_{o,z} \end{pmatrix} = {}^BR_A\;{}^AP_o$$

Position of O in A

Orientation of A in B

Coordinate Transformation



- Given:
  - Position of O in A
  - Orientation of A in B
- Find Position of O in B :

$$ {}^{B}P_{o} = {}^{A}P_{o,x}\ {}^{B}\overrightarrow{i_A} + {}^{A}P_{o,y}\ {}^{B}\overrightarrow{j_A} + {}^{A}P_{o,z}\ {}^{B}\overrightarrow{k_A} = $$

$$ \begin{pmatrix} {}^{B}i_{A,x} & {}^{B}j_{A,x} & {}^{B}k_{A,x} \\ {}^{B}i_{A,y} & {}^{B}j_{A,y} & {}^{B}k_{A,y} \\ {}^{B}i_{A,z} & {}^{B}j_{A,z} & {}^{B}k_{A,z} \end{pmatrix} \begin{pmatrix} {}^{A}P_{o,x} \\ {}^{A}P_{o,y} \\ {}^{A}P_{o,z} \end{pmatrix} = {}^{B}R_{A}\ {}^{A}P_{o} $$

- How about Orientation of O in B?

$$ {}^{B}i_{o} = {}^{A}i_{o,x}\ {}^{B}\overrightarrow{i_{A}} + {}^{A}i_{o,y}\ {}^{B}\overrightarrow{j_{A}} + {}^{A}i_{o,z}\ {}^{B}\overrightarrow{k_{A}} = \begin{pmatrix} {}^{B}i_{A,x} & {}^{B}j_{A,x} & {}^{B}k_{A,x} \\ {}^{B}i_{A,y} & {}^{B}j_{A,y} & {}^{B}k_{A,y} \\ {}^{B}i_{A,z} & {}^{B}j_{A,z} & {}^{B}k_{A,z} \end{pmatrix} \begin{pmatrix} {}^{A}i_{o,x} \\ {}^{A}i_{o,y} \\ {}^{A}i_{o,z} \end{pmatrix} = {}^{B}R_{A}\ {}^{A}i_{o} $$

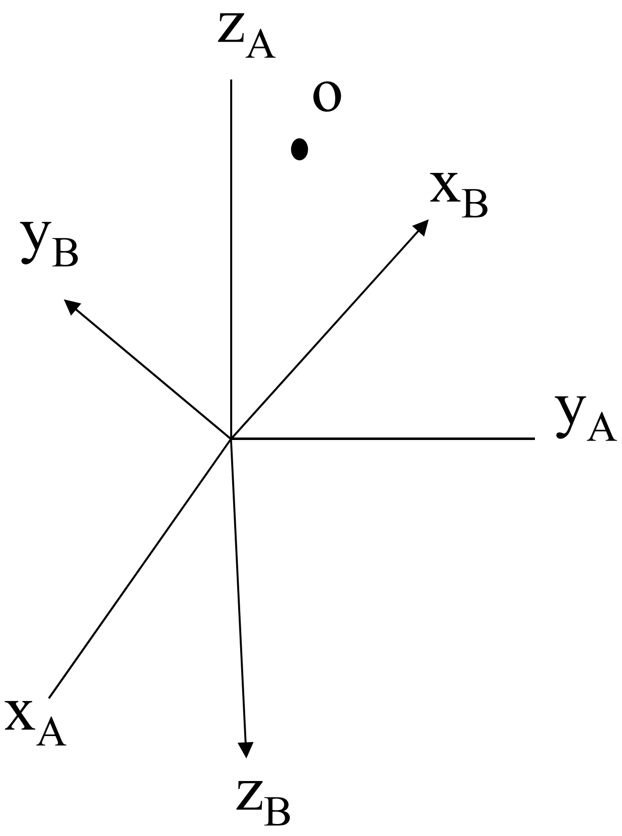
$$ {}^{B}j_{o} = \begin{pmatrix} {}^{B}i_{A,x} & {}^{B}j_{A,x} & {}^{B}k_{A,x} \\ {}^{B}i_{A,y} & {}^{B}j_{A,y} & {}^{B}k_{A,y} \\ {}^{B}i_{A,z} & {}^{B}j_{A,z} & {}^{B}k_{A,z} \end{pmatrix} \begin{pmatrix} {}^{A}j_{o,x} \\ {}^{A}j_{o,y} \\ {}^{A}j_{o,z} \end{pmatrix} = {}^{B}R_{A}\ {}^{A}j_{o} $$

$$ {}^{B}k_{o} = \begin{pmatrix} {}^{B}i_{A,x} & {}^{B}j_{A,x} & {}^{B}k_{A,x} \\ {}^{B}i_{A,y} & {}^{B}j_{A,y} & {}^{B}k_{A,y} \\ {}^{B}i_{A,z} & {}^{B}j_{A,z} & {}^{B}k_{A,z} \end{pmatrix} \begin{pmatrix} {}^{A}k_{o,x} \\ {}^{A}k_{o,y} \\ {}^{A}k_{o,z} \end{pmatrix} = {}^{B}R_{A}\ {}^{A}k_{o} $$

$$ \begin{pmatrix} {}^{B}i_{o} & {}^{B}j_{o} & {}^{B}k_{o} \end{pmatrix} = \begin{pmatrix} {}^{B}i_{A,x} & {}^{B}j_{A,x} & {}^{B}k_{A,x} \\ {}^{B}i_{A,y} & {}^{B}j_{A,y} & {}^{B}k_{A,y} \\ {}^{B}i_{A,z} & {}^{B}j_{A,z} & {}^{B}k_{A,z} \end{pmatrix} \begin{pmatrix} {}^{A}i_{o} & {}^{A}j_{o} & {}^{A}k_{o} \end{pmatrix} $$

$$ {}^{B}R_{o} = {}^{B}R_{A}\ {}^{A}R_{o} $$

The other way around: Given O in B, Find O in A



Coordinate Transformations between Frames with Non-Coincident Origins

Generalizing – a point, O, with coordinates in Frame B can be transformed to coordinates in Frame A if the relative orientation between A & B ($^{A}R_{B}$) is known

"chain rule"

Coordinates of O in B

$$^{A}P_{o} = {}^{A}R_{B}\,{}^{B}P_{o}$$

Is this always true?

Coordinates of O in A

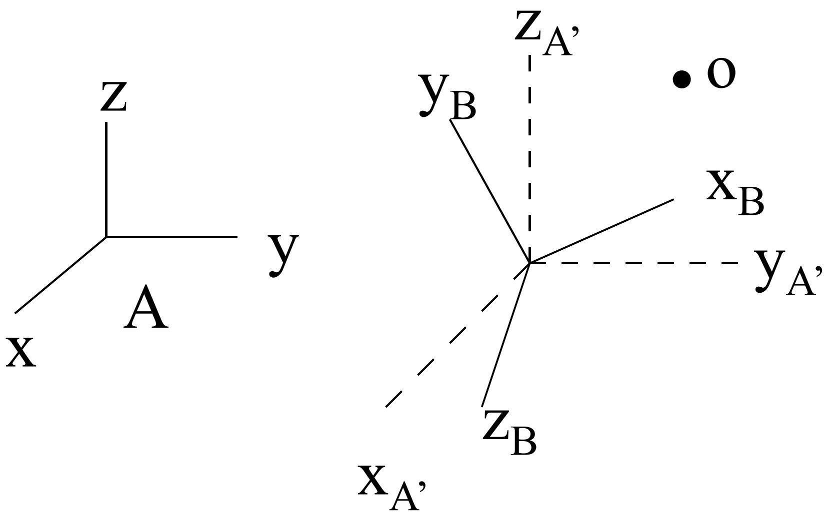
Note Assumption : – origins of Frames A & B are coincident
– only difference is in orientation

Why?

Coordinate Transformations between Frames with Non-Coincident Origins

What if the two origins are not coincident?

First: translate A to B (A becomes A')



A' = frame parallel to A
(only different origins)

$$^{A'}P_o = {}^{A'}R_B \, {}^{B}P_o \qquad \text{(from before)}$$

Coordinate Transformations between Frames with Non-Coincident Origins

$$ {}^{A}P_{o} = {}^{A}P_{B} + {}^{A'}P_{o} \Big\} $$ can only add two position vectors if they are expressed in Frames that are parallel to each other

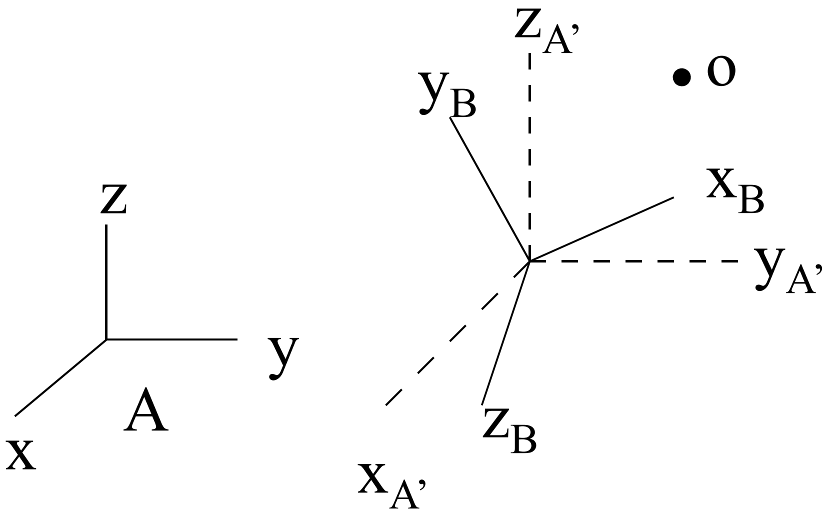
A is parallel to A' – so that adding respective coordinates make sense (axes directions are same)

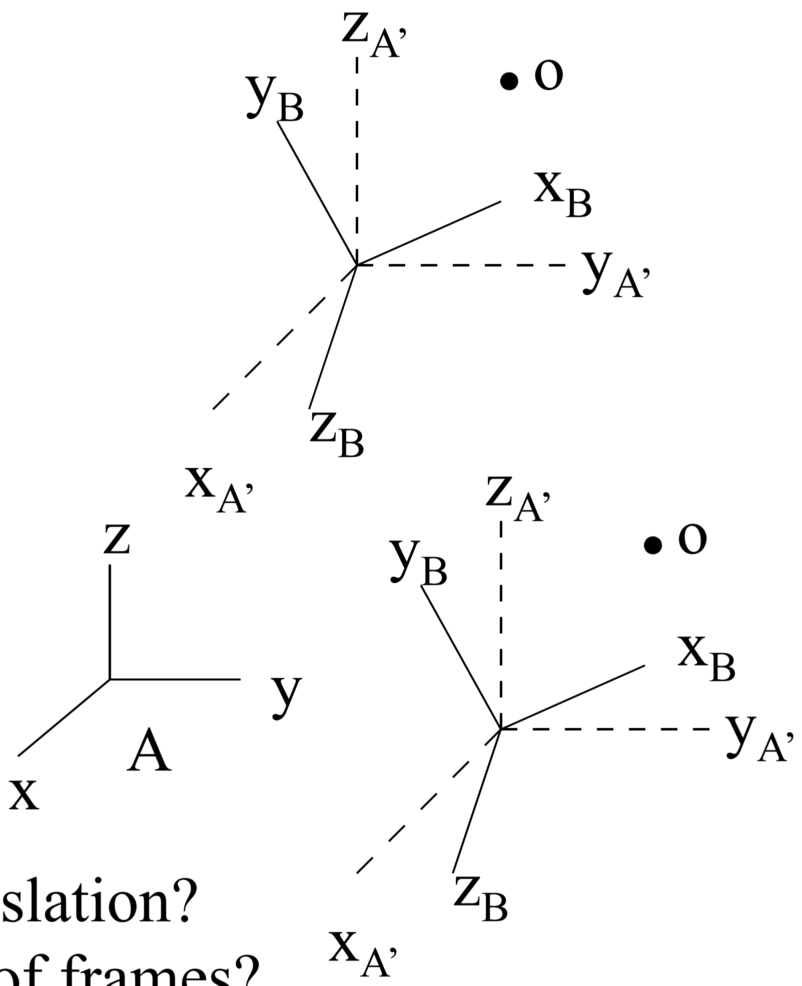
Therefore

$$ {}^{A}P_{o} = \underbrace{{}^{A}P_{B} + \underbrace{{}^{A}R_{B}\;{}^{B}P_{o}}_{\text{Transforming O to a frame // to A}}}_{\text{Then adding the relative displacement of A \& B}} $$

since ${}^{A}R_{B} = {}^{A'}R_{B}$ because A is // A'

How about Orientation Transformations?





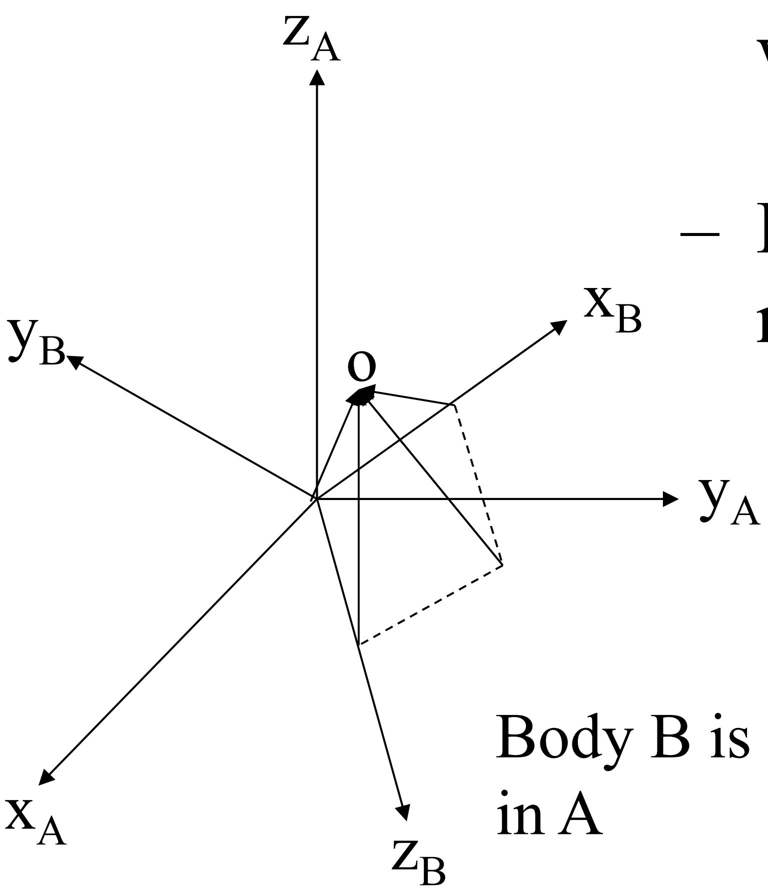
Does orientation of B change during translation?

Does orientation depend on the position of frames?

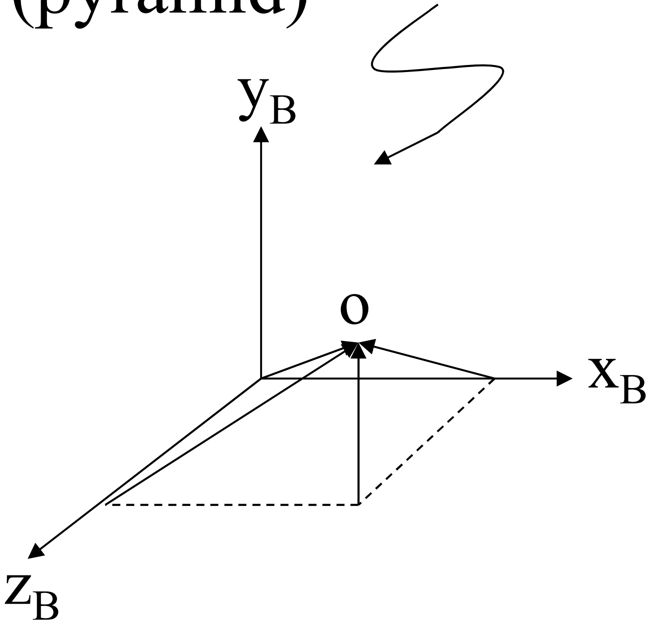
Duality (of Coordinate Transformation) with Rigid Body Motion

- Imagine a Rigid Body (B=Pyramid) whose apex is at pt. O

- Frame B is attached to the rigid body (pyramid)

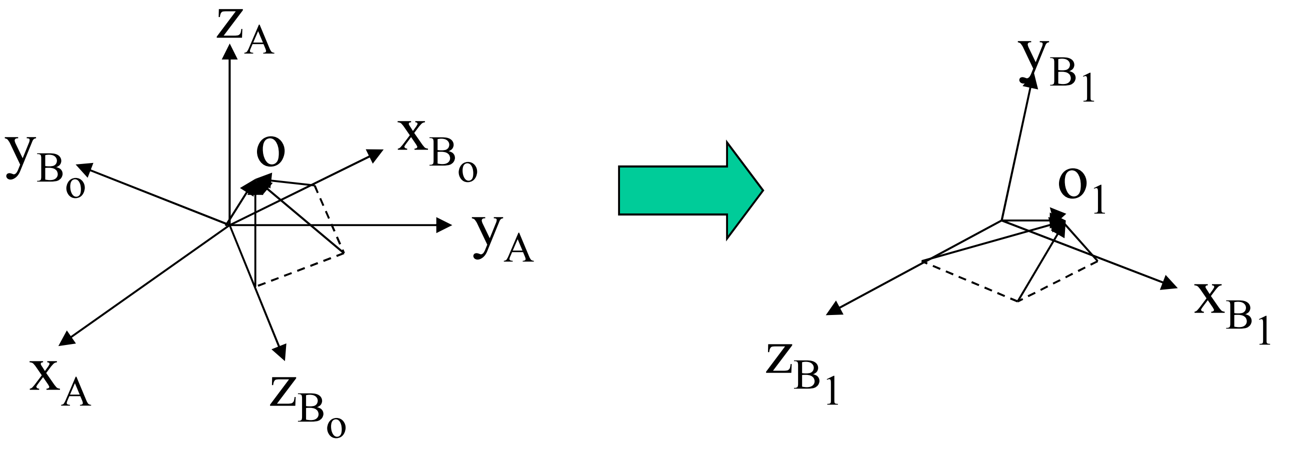


Body B is initially in A



Duality (of Coordinate Transformation) with Rigid Body Motion

- The body B is initially at $^{A}R_{Bo}$ and $^{A}P_{o} = {}^{A}R_{Bo}\,{}^{B}P_{o}$
- Body B undergoes a motion (B_0 goes to B_1) such that the new Position & Orientation of B are $^{A}R_{B_1}$ & $^{A}P_{B_1}$



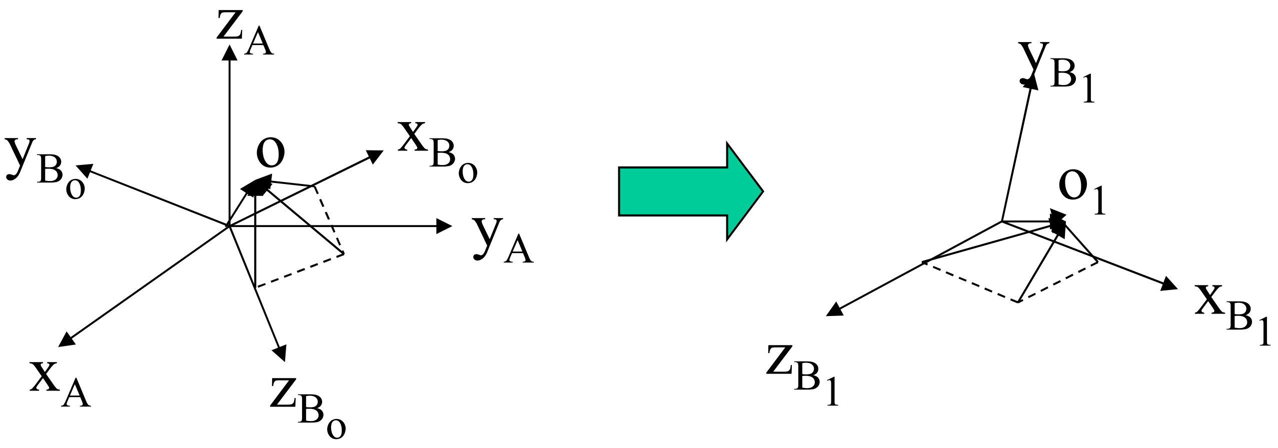
- New Coordinates of O in A, $^{A}P_{O_1}$ can be found :

$$^{A}P_{o_1} = {}^{A}P_{B_1} + {}^{A}R_{B_1}\,{}^{B_1}P_{o1}$$

Translation from A to B1

Rotations from A to B1

How about the New Orientation of B?



- New Orientation of B in A, ${}^{A}R_{B_1}$ can be found
- :

$$ {}^{A}R_{B_1} = {}^{A}R_{B_o} \, {}^{B_0}R_{B1} $$

Rotations from A to B0

Rotations from B0 to B1

"Chain Rule"

Duality in meanings of

$$ {}^{A}P_{B_1} \text{ and } {}^{A}R_{B_1} $$

- Coordinates in B (initially equal to A)
    - New coordinates in A

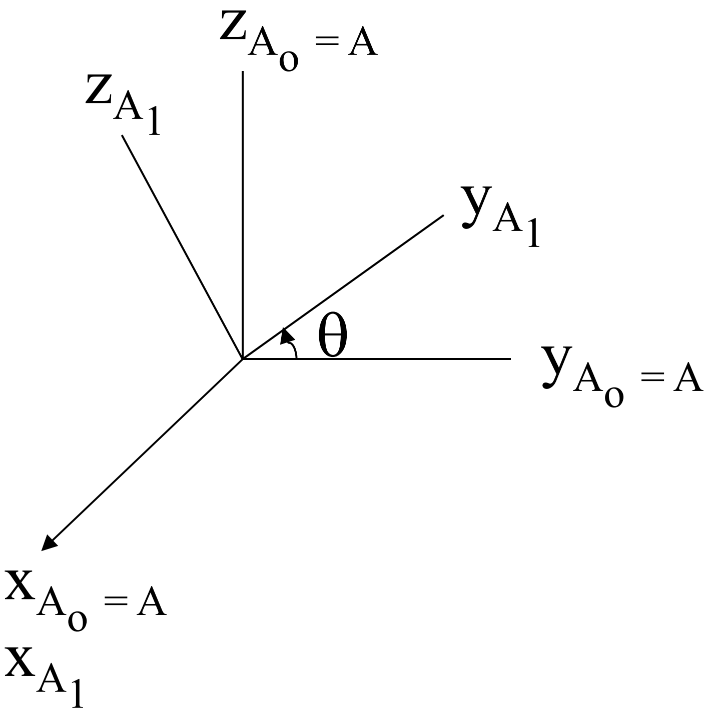
- B initially in A
    - Rigid body motion from A to B

Fundamental Motions

- Rotation about
  - X, Y, and Z axes
- Translation along
  - X, Y, and Z axes

Is sequence of motions Important?

Rotational Motion



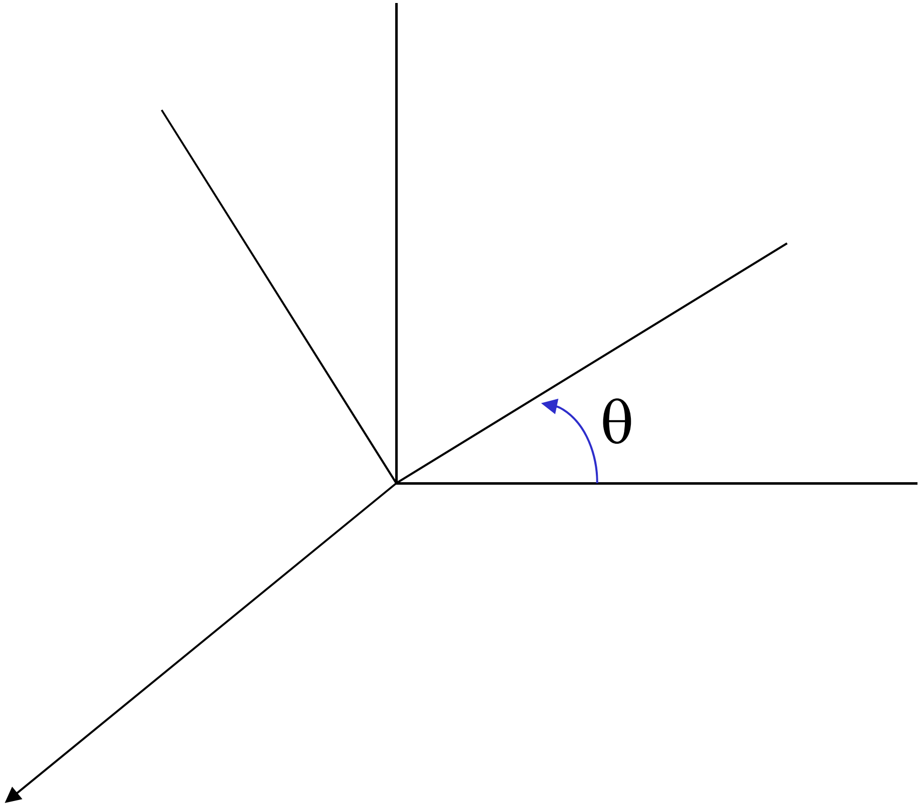
- Initially, Body is at A_o
- A undergoes a Rotation about x_{Ao} by θ & A is now at A_1 after the rotation

$$^{Ao}R_{A1} = \text{Rot}\,(x,\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Similarly, a Rotation about y by θ :

$$^{Ao}R_{A1} = \text{Rot}\,(y,\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

Rotational Motion

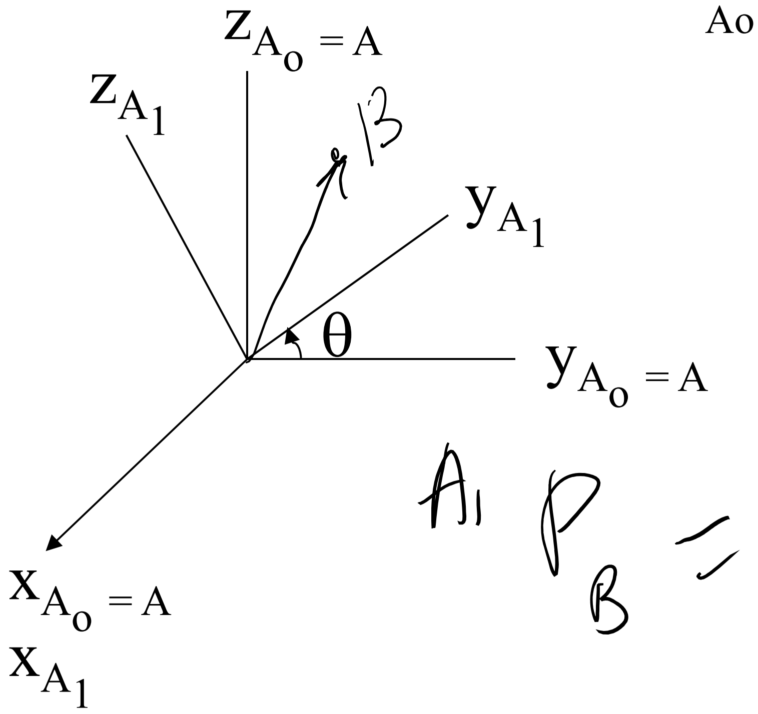


Rotational Motion

Similarly, a rotation about z by θ :

$$^{Ao}R_{A_1} = \mathrm{Rot}\,(z,\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Initially, Body is at A_o
- A undergoes a Rotation about x_{Ao} by θ & A is now at A_1 after the rotation



$$ {}^{Ao}R_{A1} = \mathrm{Rot}\,(x,\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} $$

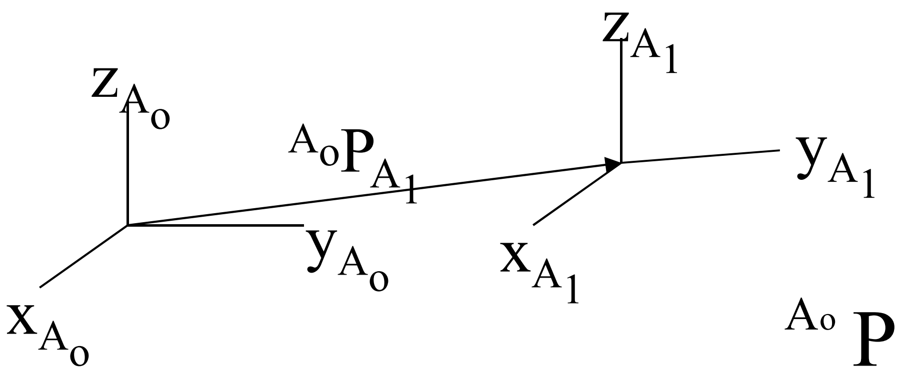
$$ {}^{A_1}P_B = \left({}^{A_1}x_{A_0} \quad {}^{A_1}y_{A_0} \quad {}^{A_1}z_{A_0} \right) {}^{A_0}P_B $$

$$ = {}^{A_1}R_{A_0}\, {}^{A_0}P_B = {}^{A_0}R^T_{A_1}\, {}^{A_0}P_B $$

New position of B in A

Translational Motion

Translation – Body A undergoes pure translation (only)

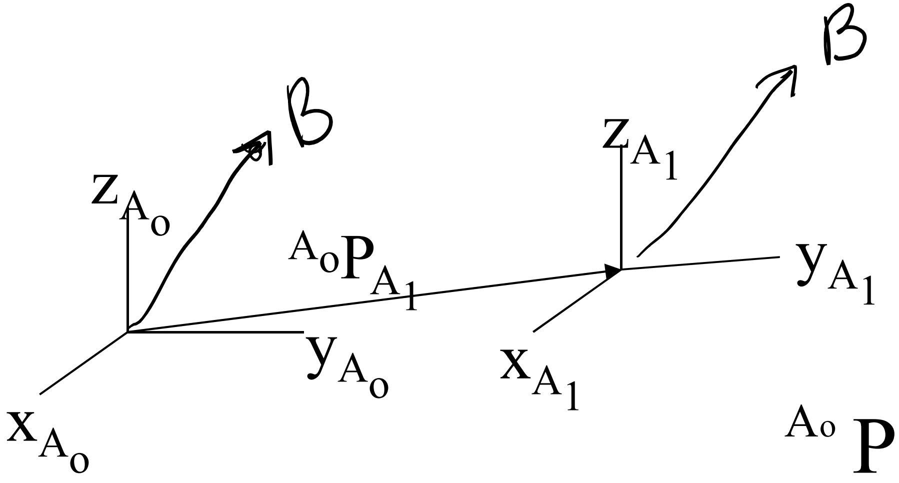


$$^{A_o}P_{A_1} = \begin{pmatrix} ^{A_o}P_{A1,x} \\ ^{A_o}P_{A1,y} \\ ^{A_o}P_{A1,z} \end{pmatrix}$$

Coordinates of A_1 in A_o

Translational Motion

Translation – Body A undergoes pure translation (only)



Coordinates of A_1 in A_o

$$
{}^{A_o}P_{A_1} = \begin{pmatrix} {}^{A_o}P_{A_{1,x}} \\ {}^{A_o}P_{A_{1,y}} \\ {}^{A_o}P_{A_{1,z}} \end{pmatrix}
$$

$$
{}^{A_1}P_B = {}^{A_o}P_B + {}^{A_o}P_{A_1}
$$

Homogeneous Transformations

- created so that rotations and translations are treated uniformly. (i.e. as matrix multiplications)
- vectors have 4 components
  - ~ scaling factor = 0 (free vectors, unit vectors along axis, orientation vectors)
  - ~ scaling factor = 1 (position vectors)

$$^{A}P_{B} = \begin{bmatrix} ^{A}P_{Bx} \\ ^{A}P_{By} \\ ^{A}P_{Bz} \\ 1 \end{bmatrix}$$

position

$$^{A}i_{B} = \begin{bmatrix} ^{A}i_{Bx} \\ ^{A}i_{By} \\ ^{A}i_{Bz} \\ \phi \end{bmatrix}$$

; similarly with j & k

Orientation (free)

Homogeneous Transformations

3×3 3×1

$$\left[\begin{array}{c} {}^{A}P_{C} \\ \hdashline 1 \end{array}\right] = \left[\begin{array}{c:c} {}^{A}R_{B} & {}^{A}P_{B} \\ \hdashline 0 & 1 \end{array}\right] \left[\begin{array}{c} {}^{B}P_{C} \\ \hdashline 1 \end{array}\right]$$

1×1

1×3

$$\rightsquigarrow \quad {}^{A}P_{C} = {}^{A}R_{B}\,{}^{B}P_{C} + {}^{A}P_{B}$$

$${}^{A}T_{B} = \left[\begin{array}{c:c} {}^{A}R_{B} & {}^{A}P_{B} \\ \hdashline 0 & 1 \end{array}\right] \in \Re^{4\times 4}$$

= homogeneous transformation matrix

= describes the position and orientation of frame B in frame A

Chain rule :

$${}^{A}T_{C} = {}^{A}T_{B}\,{}^{B}T_{C}$$

Inverse of a Homogeneous Transformation Matrix

Given ${}^{A}T_{B}$, Find ${}^{B}T_{A}$

$$ {}^{A}T_{B}\,{}^{B}T_{A} = I $$

$$\left[\begin{array}{c|c} {}^{A}R_{B} & {}^{A}P_{B} \\ \hline \underline{0} & 1 \end{array}\right]\left[\begin{array}{c|c} X & Y \\ \hline \underline{0} & 1 \end{array}\right] = \left[\begin{array}{c|c} I & \underline{0} \\ \hline \underline{0} & I \end{array}\right]$$

Block matrix operations

$$ {}^{A}R_{B}\,X = I \quad \rightsquigarrow \quad X = {}^{A}R_{B}^{-1} = {}^{A}R_{B}^{T} = {}^{B}R_{A} $$

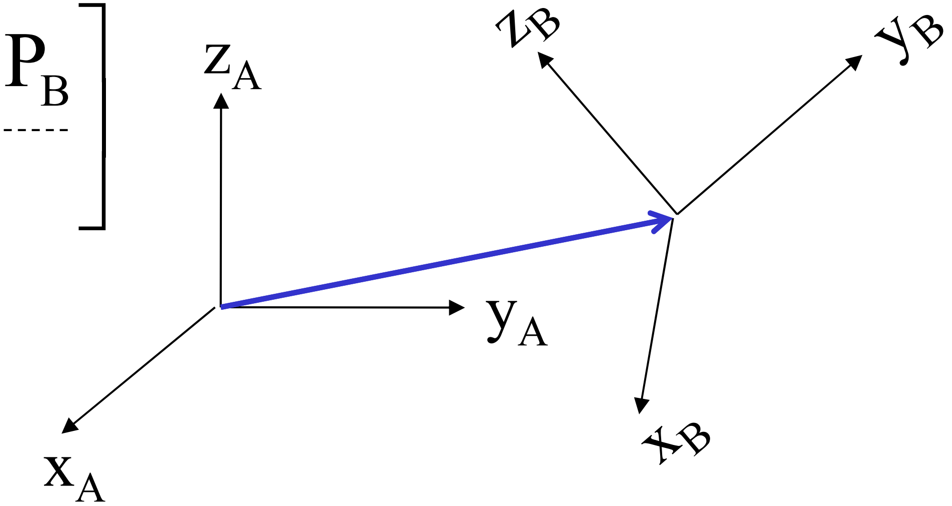
$$ {}^{A}R_{B}\,Y + {}^{A}P_{B} = \underline{0} $$

$$ {}^{A}R_{B}\,Y = -{}^{A}P_{B} $$

$$ Y = -{}^{A}R_{B}^{-1}\,{}^{A}P_{B} = -{}^{A}R_{B}^{T}\,{}^{A}P_{B} $$

Inverse of a Homogeneous Transformation Matrix

$$\therefore {}^{A}T_{B}^{-1} = \left[\begin{array}{c:c} {}^{A}R_{B}^{T} & -{}^{A}R_{B}^{T}\ {}^{A}P_{B} \\ \hdashline \underline{0} & 1 \end{array}\right]$$

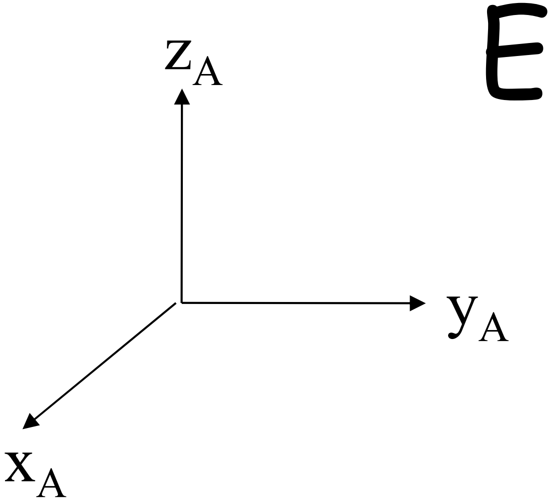


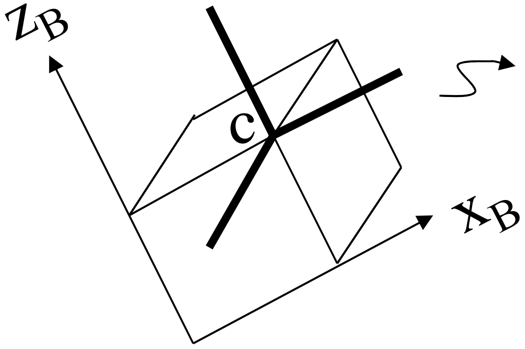
Note that

$$(\,\underline{i}\ \ \underline{j}\ \ \underline{k}\,)^{T} = {}^{A}R_{B}^{T}$$

$$\begin{aligned} {}^{B}P_{A} = -{}^{A}R_{B}^{T}\ {}^{A}P_{B} = & -\underline{i} \bullet {}^{A}P_{B} \\ & -\underline{j} \bullet {}^{A}P_{B} \\ & -\underline{k} \bullet {}^{A}P_{B} \end{aligned}$$

Example





cube of unit length

Given : Initial Position & Orientation of cube specified by ${}^{A}T_{B}$

Find : new coordinates of pt. c (corner of cube) after the cube is rotated by θ about z_A

Answer :

$$ {}^{A}P_{C} = \underbrace{\mathrm{Rot}(z, 90^{\circ})\ {}^{A}T_{B}}_{{}^{A}P_{B'}} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{{}^{B'}P_{C} = {}^{B}P_{C}} $$

since C is attached rigidly to B

Two Rules

- Given Bodies A and B, with ${}^{A}T_{B}$
- Body B moves, new position and orientation is ${}^{A}T_{B1}$
  - with respect to A : Pre-multiplication
    - ${}^{A}T_{B1} = \text{Motion } (4\text{x}4)\ {}^{A}T_{B}$
  - with respect to B (body itself): Post multiplication
    - ${}^{A}T_{B1} = {}^{A}T_{B}\ \text{Motion } (4\text{x}4)$

Why?

Other Orientation Representations

- Relative orientation between two bodies A and B can **always** be described by a sequence of 3 rotations about adjacent axes that are orthogonal to each other
- The 3 angles of rotation are used to represent orientation, together with their meanings (axes of rotation)

$$^{A}R_{B} = Rot(x,d)Rot(y,e)Rot(x,f)$$

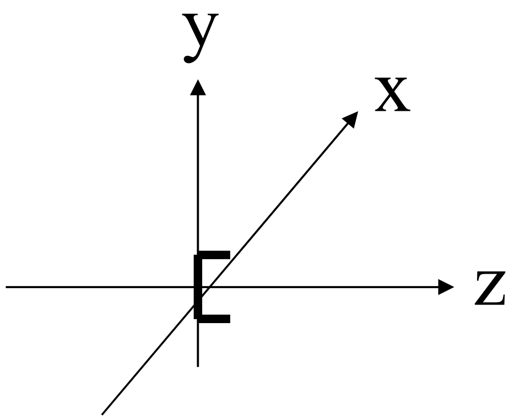
$$^{A}R_{B} = Rot(x,a)Rot(y,b)Rot(z,c)$$

$$^{A}R_{B} = Rot(z,p)Rot(y,q)Rot(z,r)$$

Many possibilities

- Given a definition or sequence of axes of rotations, are the three angles unique?

ROLL - PITCH - YAW Angles (ϕ, θ, φ)



$$ {}^{A}R_{B} = \mathrm{Rot}(z, \phi)\ \mathrm{Rot}(y, \theta)\ \mathrm{Rot}(x, \varphi) $$

$$ \underset{\text{Roll}}{\uparrow} \qquad \underset{\text{Pitch}}{\uparrow} \qquad \underset{\text{Yaw}}{\uparrow} $$

$$ = \begin{pmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta\sin\varphi - \sin\phi\cos\varphi & \cos\phi\sin\theta\cos\varphi + \sin\phi\sin\varphi \\ \sin\phi\cos\theta & \sin\phi\sin\theta\sin\varphi + \cos\phi\cos\varphi & \sin\phi\sin\theta\cos\varphi - \cos\phi\sin\varphi \\ -\sin\theta & \cos\theta\sin\varphi & \cos\theta\cos\varphi \end{pmatrix} $$

Forward Transformation:

Given roll (ϕ), pitch (θ), and yaw (φ) angles, find ${}^{A}R_{B}$

Inverse Transformation

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \begin{pmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta\sin\varphi - \sin\phi\cos\varphi & \cos\phi\sin\theta\cos\varphi + \sin\phi\sin\varphi \\ \sin\phi\cos\theta & \sin\phi\sin\theta\sin\varphi + \cos\phi\cos\varphi & \sin\phi\sin\theta\cos\varphi - \cos\phi\sin\varphi \\ -\sin\theta & \cos\theta\sin\varphi & \cos\theta\cos\varphi \end{pmatrix}$$

↑

given

Find the roll (ϕ), pitch (θ), and yaw (φ) angles

$$n_x = \cos\phi\ \cos\theta \qquad o_z = \cos\theta\ \sin\varphi$$

$$n_y = \sin\phi\ \cos\theta \qquad a_z = \cos\theta\ \cos\varphi$$

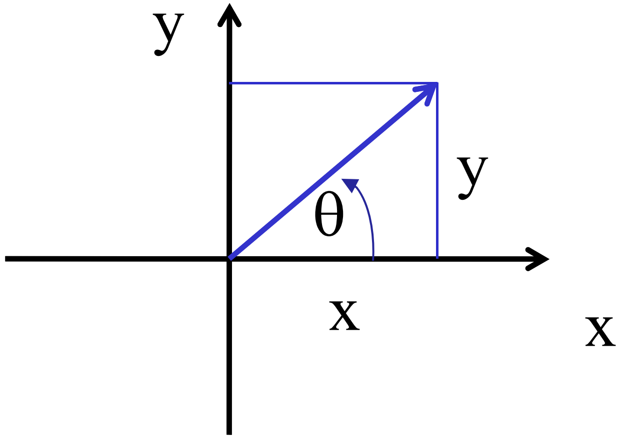
$$\text{if} \quad |n_z| \neq 1 \quad \left.\right\} \phi = \operatorname{atan2}\left(\frac{n_y/\cos\theta}{n_x/\cos\theta}\right) \qquad \varphi = \operatorname{atan2}\left(\frac{o_z}{a_z}\right) \qquad \theta = \operatorname{atan2}\left(\frac{-n_z}{\pm\sqrt{n_x^2+n_y^2}}\right)$$

2 solutions:

$$\left\{ \begin{array}{lll} \phi = \operatorname{atan2}\left(\frac{n_y}{n_x}\right) & \varphi = \operatorname{atan2}\left(\frac{o_z}{a_z}\right) & \theta = \operatorname{atan2}\left(\frac{-n_z}{+\sqrt{n_x^2+n_y^2}}\right) \\ \phi = \operatorname{atan2}\left(\frac{-n_y}{-n_x}\right) & \varphi = \operatorname{atan2}\left(\frac{-o_z}{-a_z}\right) & \theta = \operatorname{atan2}\left(\frac{-n_z}{-\sqrt{n_x^2+n_y^2}}\right) \end{array} \right.$$

Atan2 function

- Gives a unique answer
- Atan2(x,y) in software run-time libraries
- Atan2(y,x) or Atan2 (y/x) in my notes



$$\theta = \text{atan}\,2\left(\frac{y}{x}\right)$$

What if $n_z = \pm 1$?

- Mathematical singularity
  - Boundary between solution regions

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \begin{pmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta\sin\varphi\text{-}\sin\phi\cos\varphi & \cos\phi\sin\theta\cos\varphi+\sin\phi\sin\varphi \\ \sin\phi\cos\theta & \sin\phi\sin\theta\sin\varphi+\cos\phi\cos\varphi & \sin\phi\sin\theta\cos\varphi\text{-}\cos\phi\sin\varphi \\ \text{-}\sin\theta & \cos\theta\sin\varphi & \cos\theta\cos\varphi \end{pmatrix} =$$

$$\theta = 90 \qquad \begin{pmatrix} 0 & -\sin(\phi-\varphi) & \cos(\phi-\varphi) \\ 0 & \cos(\phi-\varphi) & \sin(\phi-\varphi) \\ -1 & 0 & 0 \end{pmatrix} \qquad \phi-\varphi = a\tan 2\left(\frac{-o_x}{o_y}\right)$$

$$or \quad a\tan 2\left(\frac{a_y}{a_x}\right)$$

$$\theta = 270 \qquad \begin{pmatrix} 0 & -\sin(\phi+\varphi) & -\cos(\phi+\varphi) \\ 0 & \cos(\phi+\varphi) & -\sin(\phi+\varphi) \\ 1 & 0 & 0 \end{pmatrix} \qquad \phi+\varphi = a\tan 2\left(\frac{-o_x}{o_y}\right)$$

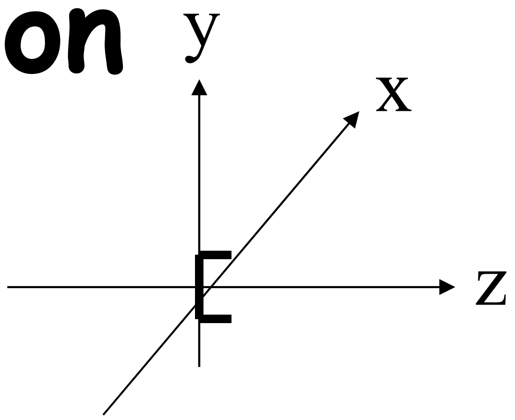
$$or \quad a\tan 2\left(\frac{-a_y}{-a_x}\right)$$

Infinite number of solutions

ϕ and φ describe the same rotation and cannot be computed separately

Mathematical Singularity of RPY representation

$$^{A}R_{B} = \text{Rot}\,(\,z,\,\underset{\substack{\uparrow\\ \text{Roll}}}{\phi}\,)\ \text{Rot}\,(\,y,\,\underset{\substack{\uparrow\\ \text{Pitch}}}{\theta}\,)\ \text{Rot}\,(\,x,\,\underset{\substack{\uparrow\\ \text{Yaw}}}{\varphi}\,)$$



$$\begin{pmatrix} 0 & -\sin(\phi-\varphi) & \cos(\phi-\varphi) \\ 0 & \cos(\phi-\varphi) & \sin(\phi-\varphi) \\ -1 & 0 & 0 \end{pmatrix}$$

At pitch = 90 deg

Roll (z) and Yaw (x) describes the same motion

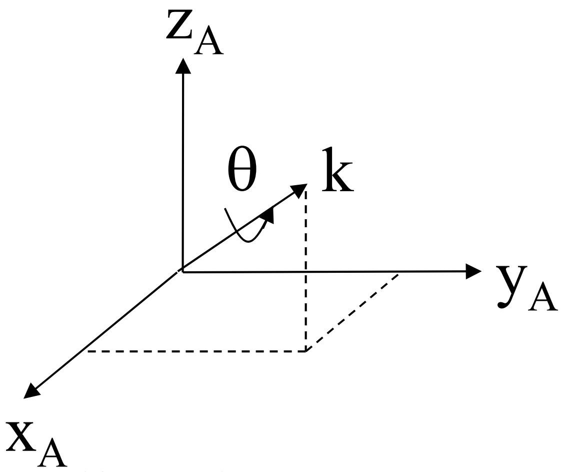
Z and X axes are aligned

Four-Parameter Representation For Orientation

- quadruple of ordered real parameters consisting of one scalar (angle of rotation) and one vector (axis of rotation)

Any rotation matrix can be represented by rotation θ of space about a fixed axis k = $\begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix}$

$^{A}R_{B} = \text{Rot}\,(\,k\,,\,\theta\,)$



A ⟶ B after rotation
K is a vector expressed in A

$$\mathrm{Rot}(k,\ \theta\ ) = \begin{pmatrix} k_x{}^2 \mathrm{vers}\theta\ + \cos\theta & k_y k_x \mathrm{vers}\theta - k_z \sin\theta & k_z k_x \mathrm{vers}\theta + k_y \sin\theta \\ k_x k_y \mathrm{vers}\theta + k_z \sin\theta & k_y{}^2 \mathrm{vers}\theta + \cos\theta & k_z k_y \mathrm{vers}\theta - k_x \sin\theta \\ k_x k_z \mathrm{vers}\theta - k_y \sin\theta & k_y k_z \mathrm{vers}\theta + k_x \sin\theta & k_z{}^2 \mathrm{vers}\theta + \cos\theta \end{pmatrix}$$

where vers $\theta = 1 - \cos\theta$

Try proving this

Hint : imagine $\vec{k}$ to be the x, y or z axis of a new frame.
Rot (k ,θ) then becomes one of the elementary rotations
[Rot (x, θ) , Rot (y, θ) or Rot (z, θ)]

Try finding the inverse transformation

Given Rotation Matrix, Find axis and angle of rotation.

Is it unique?

$Rk = k$ why?

k is an eigenvector of R corresponding to eigenvalue of 1

$$Rk = Ik \Rightarrow (R - I)k = 0$$

$$0 = R^T 0 + 0$$

$$= R^T(R - I)k + (R - I)k = (R^TR - R^T + R - I)k$$

$$= (I - R^T + R - I)k$$

$$(R - R^T)k = 0$$

$R - R^T = \hat{k}$ cross product operator of k

$$\hat{k}k = k \times k = 0$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} - \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} = \begin{pmatrix} 0 & b-d & c-g \\ d-b & 0 & f-h \\ g-c & h-f & 0 \end{pmatrix} = \hat{k} \qquad k = \begin{pmatrix} h-f \\ c-g \\ d-b \end{pmatrix}$$

$\|k\| = 2 \sin\theta$

θ = angle of rotation

Trace of $R = 1 + 2\cos\theta$