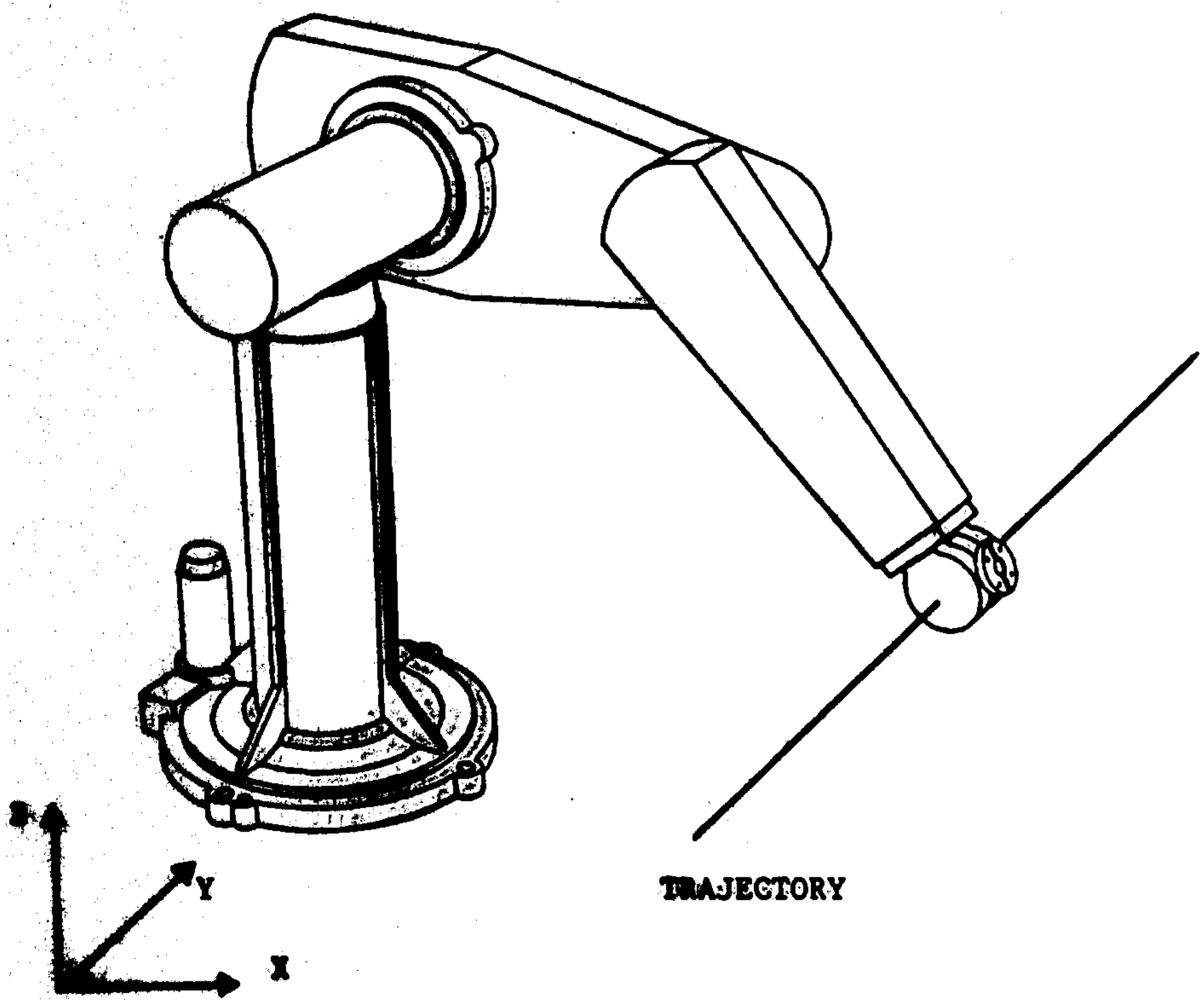
CHAPTER 2

Robot Kinematics of Position

Learning Objectives

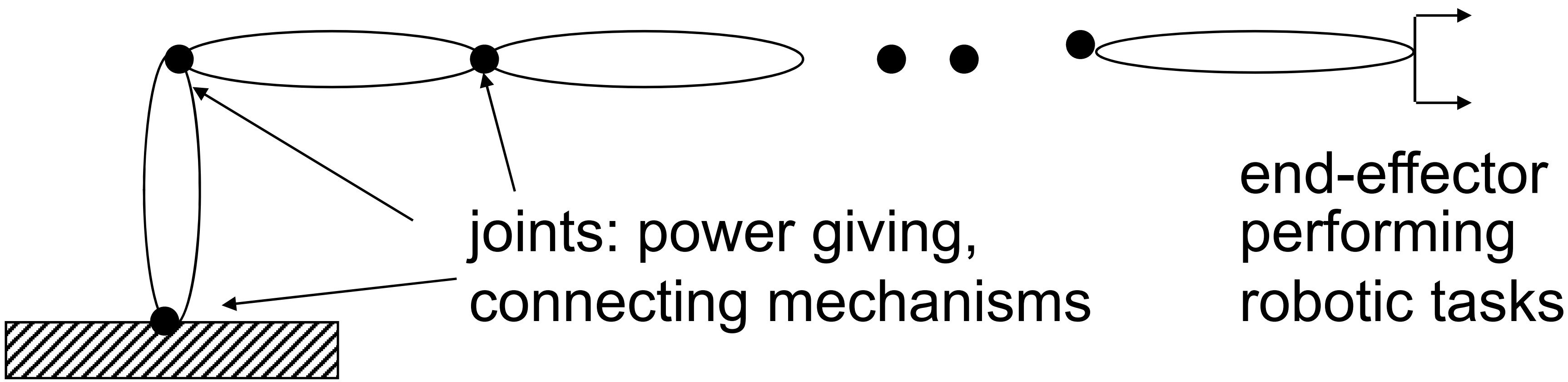
- Given a robot, derive a kinematic (analytical and geometric) model of the robot
  - Assign frames (why?)
  - Derive equations relating relative position and orientation of frames (forward and inverse equations)
    - So we know the relative position and orientation of any link with respect to any other link (including the "world", base or ref)

Robotic Manipulator



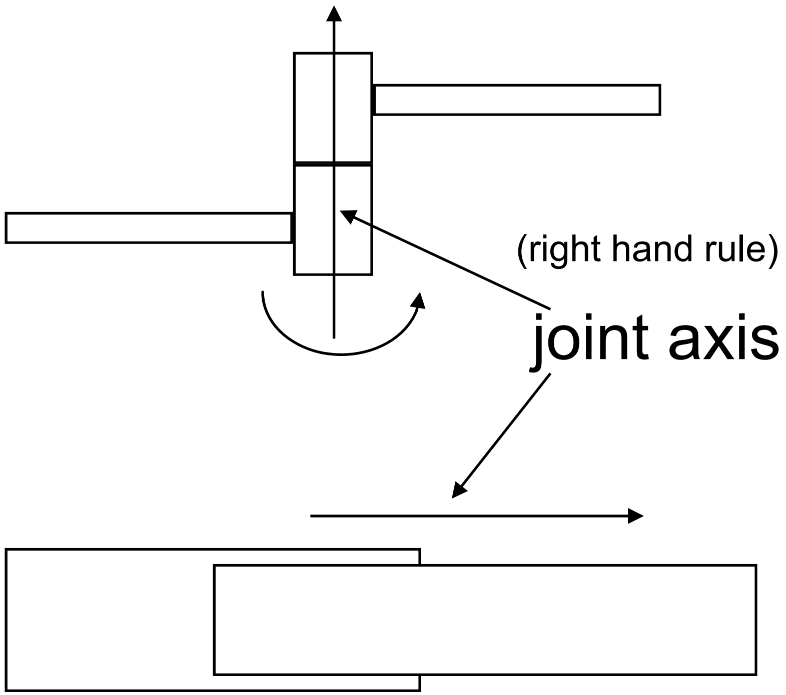
Robotic Tasks
- positioning/orienting
- force/moment exerted on environment

Chain of rigid bodies connected by joints



Robot Joints

Two Basic Types:



Rotational

Translational
(Prismatic)

Degrees-of-Freedom

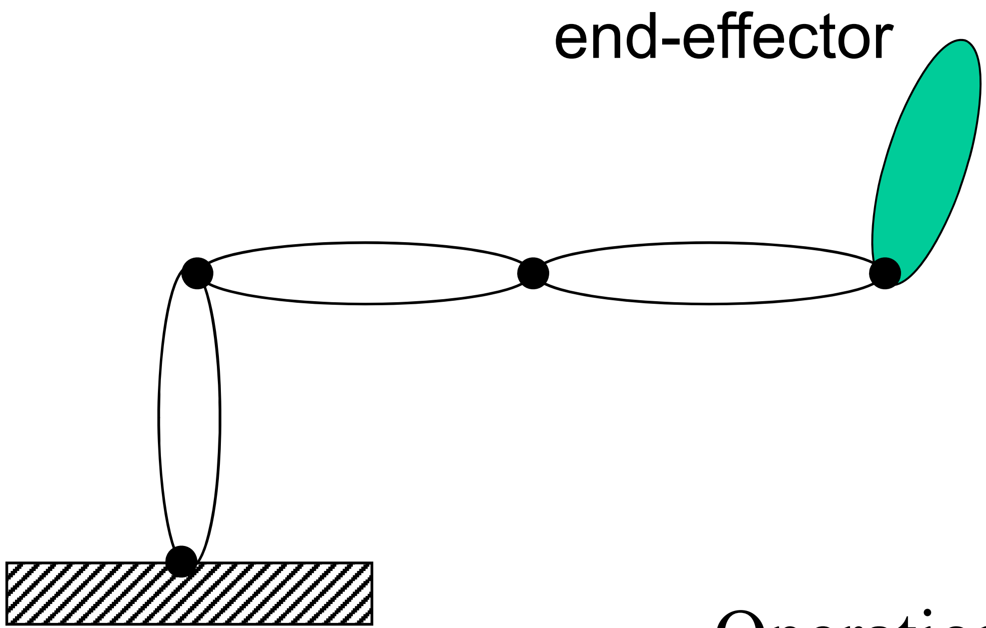
3D Space = 6 DOF → 3 position, 3 orientation

In robotics,

DOF = number of independently driven joints

As DOF ↑

- positioning accuracy ↓
- computational complexity ↑
- cost ↑
- flexibility ↑
- power transmission is more difficult



3D World Space:

m parameters

6 independent parameters

To completely specify: $m \geq 6$

Operational/Task Space:

$m_k < m$ and $m_k < 6$: subset of end-effector parameters to accomplish the task

Joint Space:

n DOF

Robot Kinematic Modeling

- Relative position and orientation of all links
- Assign frames to each link
- How many parameters are needed, in general, to describe relative position and orientation of 2 bodies?
  - 6?
  - Can we make it less?

Denavit Hartenberg Representation

- Link moves with respect to an adjacent (previous) link
- Assign frames to each link (including first [non-moving] link)
- Two rules for assigning frames to each link
  - Z axis is the axis of motion (Link i moves around or along Z_{i-1})
  - X axis is the common normal of its z axis (Z_i) and z axis of previous link (Z_{i-1})

Step 1: Number joints and links

- N joints, N moving links + 1 fixed link (base)
  - Link 0 to Link N
  - Joint 1 to Joint N
- Joint i connects Links i and i-1
  - no joint at the end of the final link (end-effector)
- Each joint provides only 1 degree-of-freedom
  - Rotation, or
  - Translation

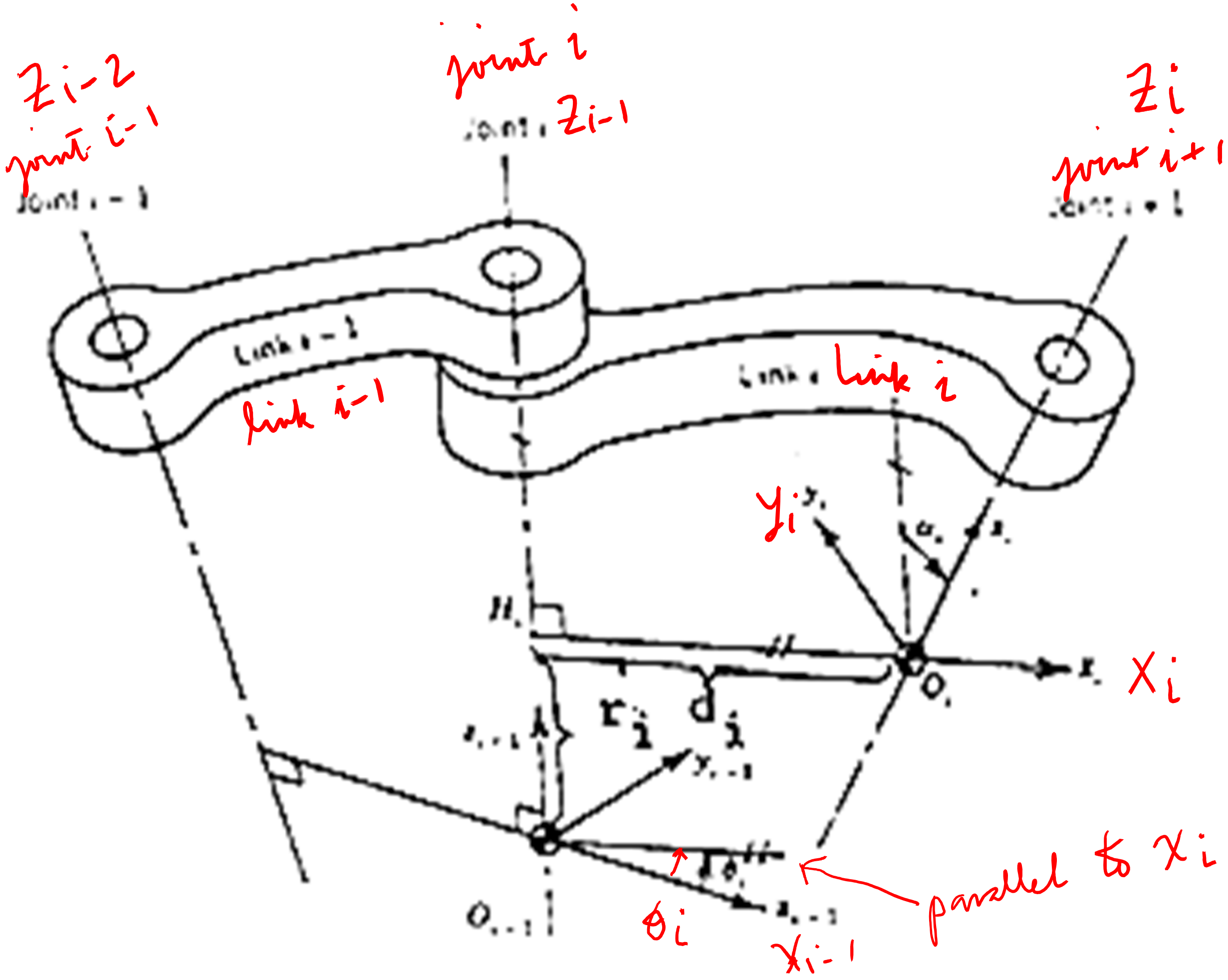
Step 2: Assign Coordinate Frames to Each Link

Assign a Cartesian coordinate frame ($O_i;x_i,y_i,z_i$) to each link (link i), as follows:

- the z_i axis is directed along the axis of motion of joint (i + 1), that is, link (i + 1) rotates about or translates along z_i;
- the x_i axis lies along the common normal from the z_{i-1} axis to the z_i axis.
  - if z_{i-1} is parallel to z_i,
    - then x_i is specified arbitrarily, subject only to x_i being perpendicular to z_i (Cartesian Coordinate Frame)
- the y_i axis completes the right-handed coordinate system

Step 2: First and Last Links

- Link 0 (Frame 0) – fixed base of robot
  - Z_0 axis has meaning – link 1 joint axis – 1st axis of motion
  - X_0 and Y_0 arbitrarily specified to have some physical meaning (e.g, front, back)
- Link N (Frame N) – end-effector (or "hand") – last link
  - Z_N axis arbitrarily specified to have some physical meaning (e.g., pointing direction)
  - X_N axis – specified following Denavit-Hartenberg rule (common normal between Z and previous Z)
  - Y_N axis – completes the cartesian frame (right hand rule)



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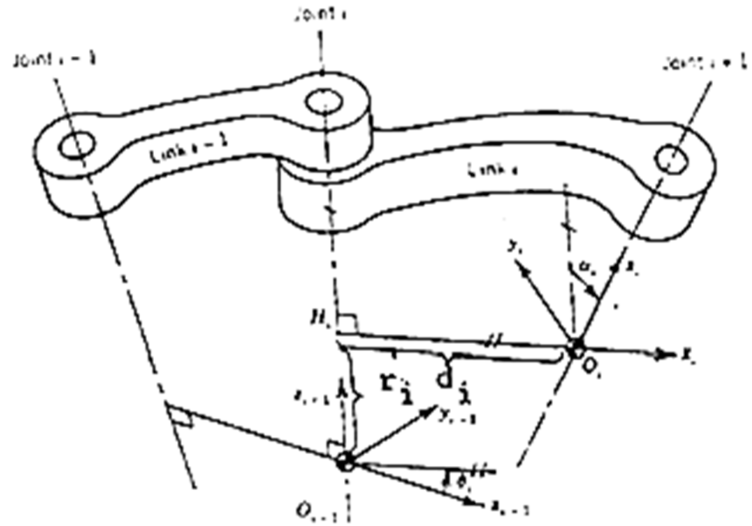
translates along z_i;

the x_i axis lies along the common normal from
the z_{i-1} axis to the z_i axis

if z_{i-1} is parallel to z_i,
then x_i is specified arbitrarily, subject only to x_i
being perpendicular to z_i

Step 3: Define Joint Coordinates (Joint Variables)



- q_i
- If joint i is rotational (link i rotating with respect to link i-1)
  - q_i is the angular displacement with respect to link i-1
- If joint i is translational (link i translating with respect to link i-1)
  - q_i is the linear displacement with respect to link i-1

The N-dimensional space defined by the joint coordinates $(q_1,\ldots,q_N)$ is called the configuration space of the N DOF mechanism.

Step 4: Identify the Link Kinematic Parameters

In general, four elementary transformations are required to relate the *i*-th coordinate frame to the (*i*-1)-th coordinate frame:

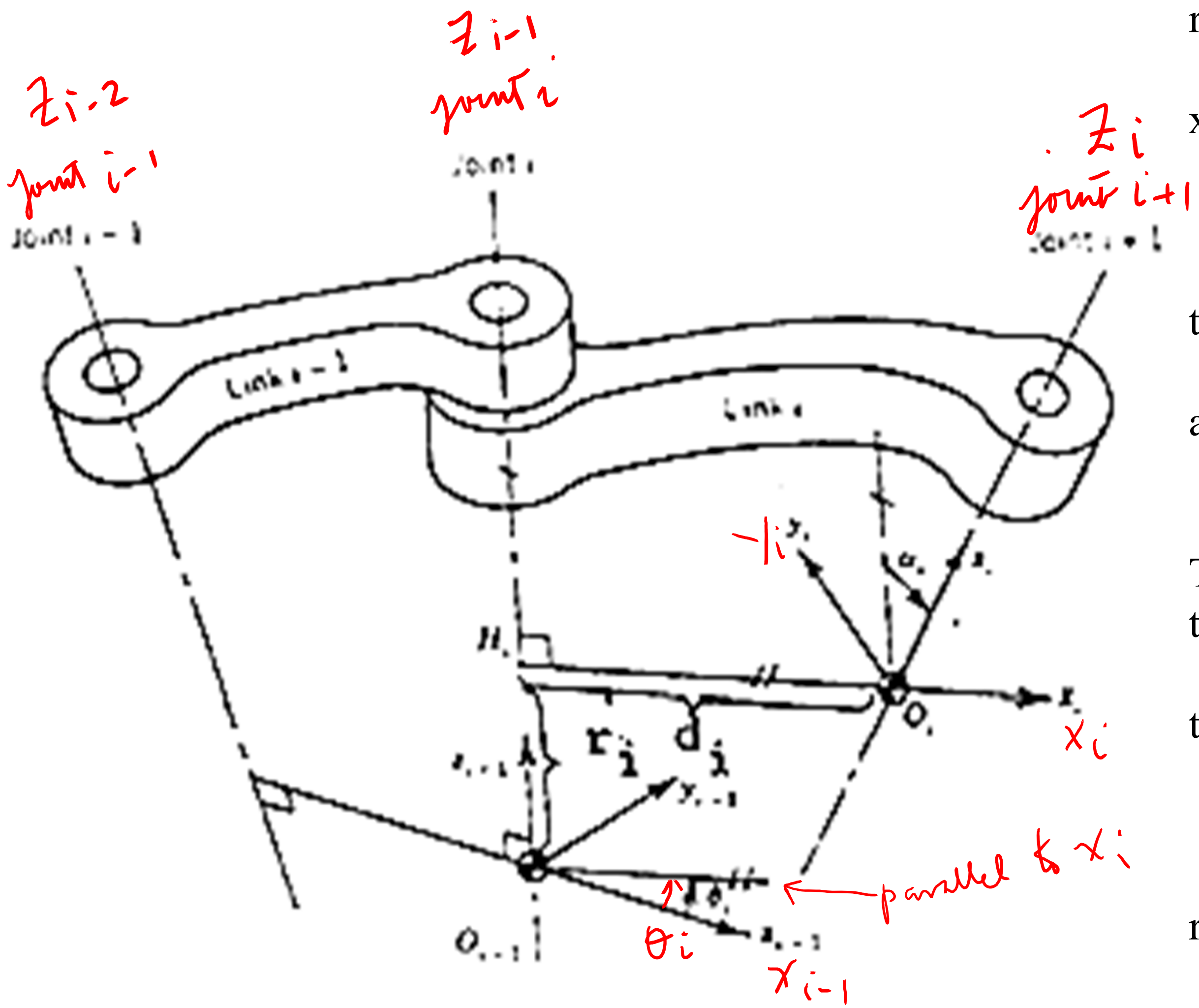
- Rotate an angle of θ_i (in the right-handed sense) about the z_{i-1} axis, so that the x_{i-1} axis is parallel to the x_i axis.
- Translate a distance of r_i along the positive direction of the z_{i-1} axis, to align the x_{i-1} axis with the x_i axis.

Step 4: Identify the Link Kinematic Parameters

- Translate a distance of d_i along the positive direction of the $x_{i-1} = x_i$ axis, to coalesce the origins O_{i-1} and O_i.
- Rotate an angle of α_i (in the right-handed sense) about the $x_{i-1} = x_i$ axis, to coalesce the two coordinate systems.

The *i*-th coordinate frame is therefore characterized by the four D-H kinematic link parameters θ_i, r_i, d_i and α_i.

The 4 Elementary Motions



Rotate an angle of θ_i (in the right-handed sense) about the z_{i-1} axis, so that the x_{i-1} axis is parallel to the x_i axis.

Translate a distance of r_i along the positive direction of the z_{i-1} axis, to align the x_{i-1} axis with the x_i axis.

Translate a distance of d_i along the positive direction of the $x_{i-1} = x_i$ axis, to coalesce the origins O_{i-1} and O_i.

Rotate an angle of α_i (in the right-handed sense) about the $x_{i-1} = x_i$ axis, to coalesce the two coordinate systems.

Step 4: Link Kinematic Parameters: $\theta i, ri, di, \alpha i$

If joint *i* is rotational, then $q_i = \theta_i$, and α_i, d_i and r_i are constant parameters which depend upon the geometric properties and configuration of link *i*. (Step 3)

If joint *i* is translational, then $q_i = r_i$, and d_i, α_i and θ_i are constant parameters which depend upon the configuration of link *i*. (Step 3)

For both rotational and translational joints, r_i and θ_i are the distance and angle between links $(i - 1)$ and i; d_i and α_i are the length and twist of link *i*.

Step 5: Define Link Transformation Matrices

The position and orientation of the *i*-th coordinate frame can be expressed in the (*i* – 1)-th coordinate frame by the following homogeneous transformation matrix:

$$^{i-1}T_i = A_i = Rot(z, \theta)\ Trans(0, 0, r_i)\ Trans(d_i, 0, 0)\ Rot(x, \alpha)$$

$$A_i(q_i) = {}^{i-1}T_i = \begin{pmatrix} \cos\theta_i & -\cos\alpha_i \sin\theta_i & \sin\alpha_i \sin\theta_i & d_i \cos\theta_i \\ \sin\theta_i & \cos\alpha_i \cos\theta_i & -\sin\alpha_i \cos\theta_i & d_i \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & r_i \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Step 6: Compute the Forward Transformation Matrix

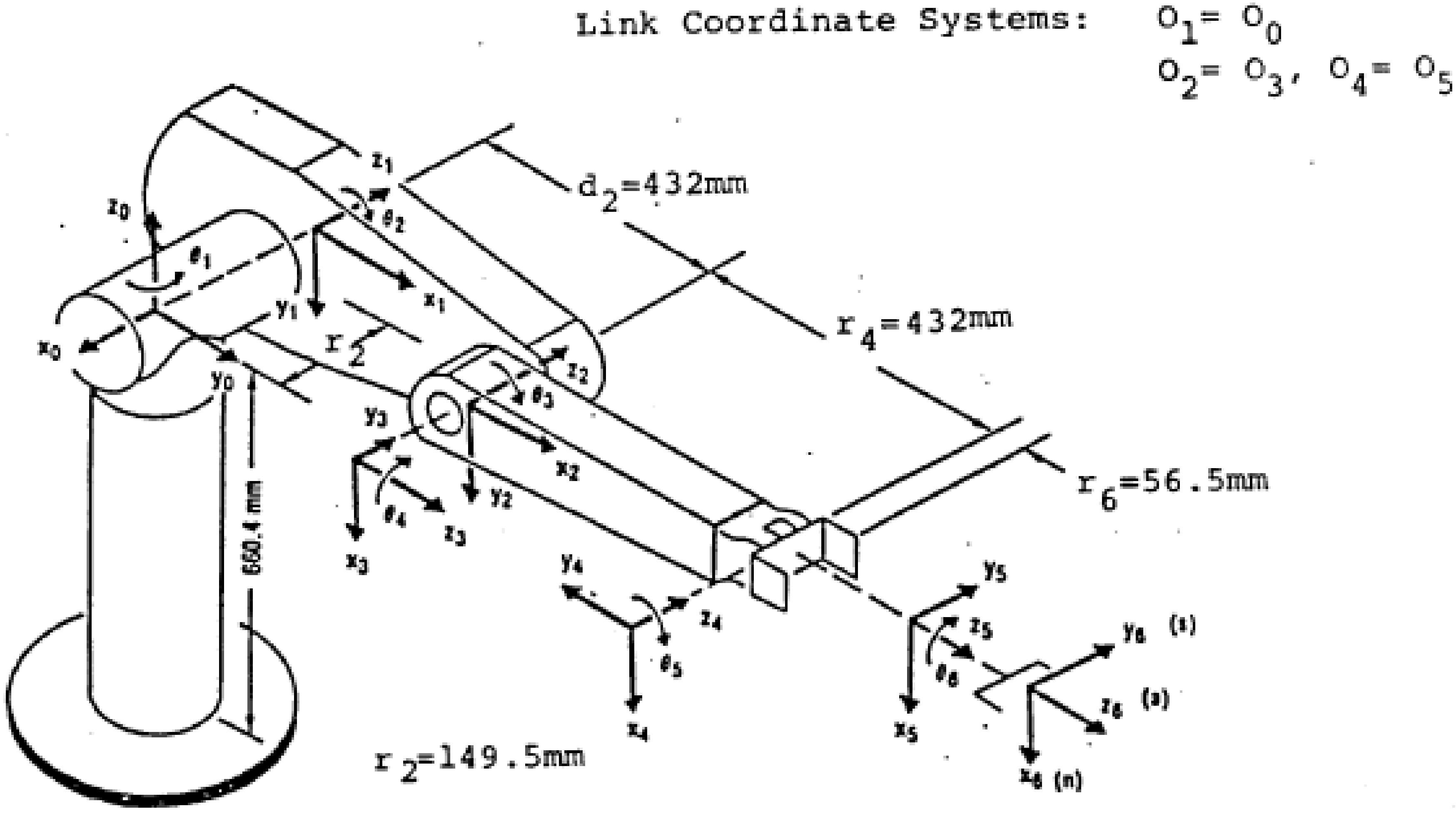
The position and orientation of the end-effector coordinate frame is expressed in the base coordinate frame by the forward transformation matrix:

$$
{}^{0}T_N(q_1, q_2, \ldots, q_N) = {}^{0}T_N = A_1 A_2 \ldots A_N = \begin{pmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

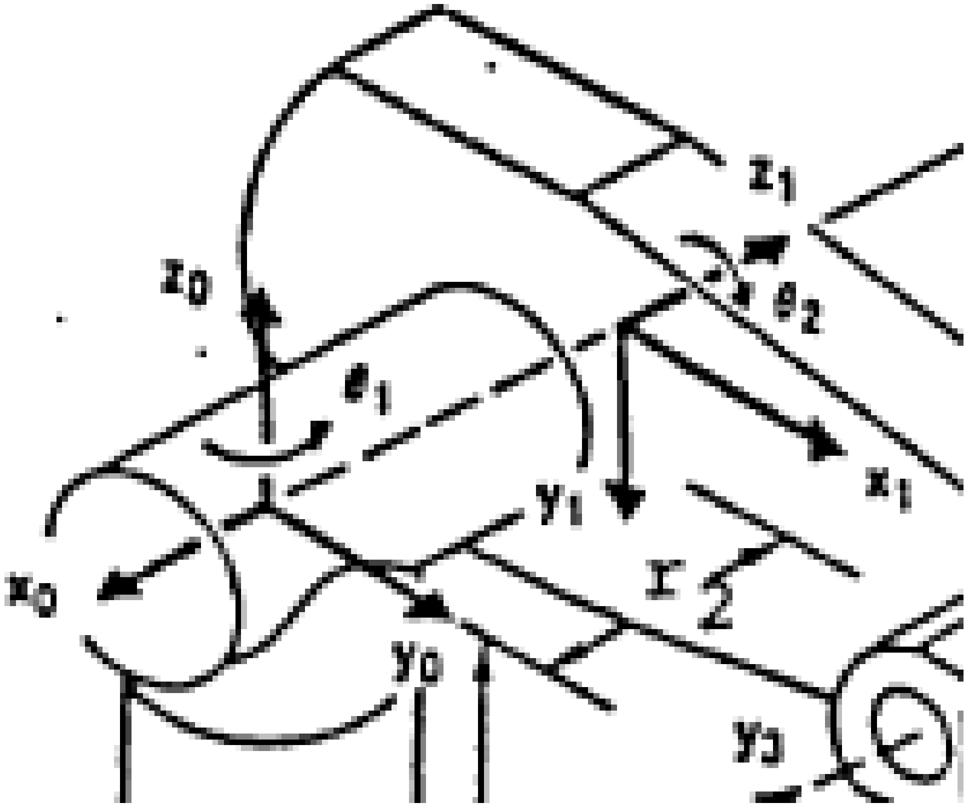
Position of any frame (link) with respect to any other frame (link) can be computed: ${}^{j}T_i \quad i, j = 1.....N$

Robot Kinematic Modeling

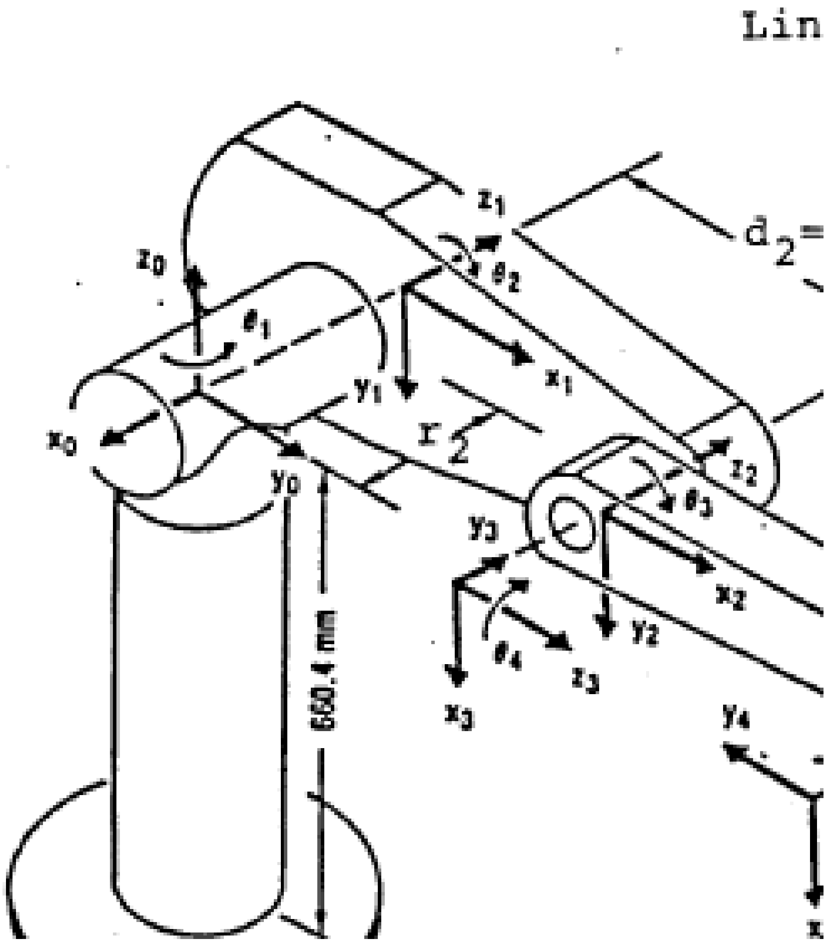
EXAMPLE 1: The Puma Robot



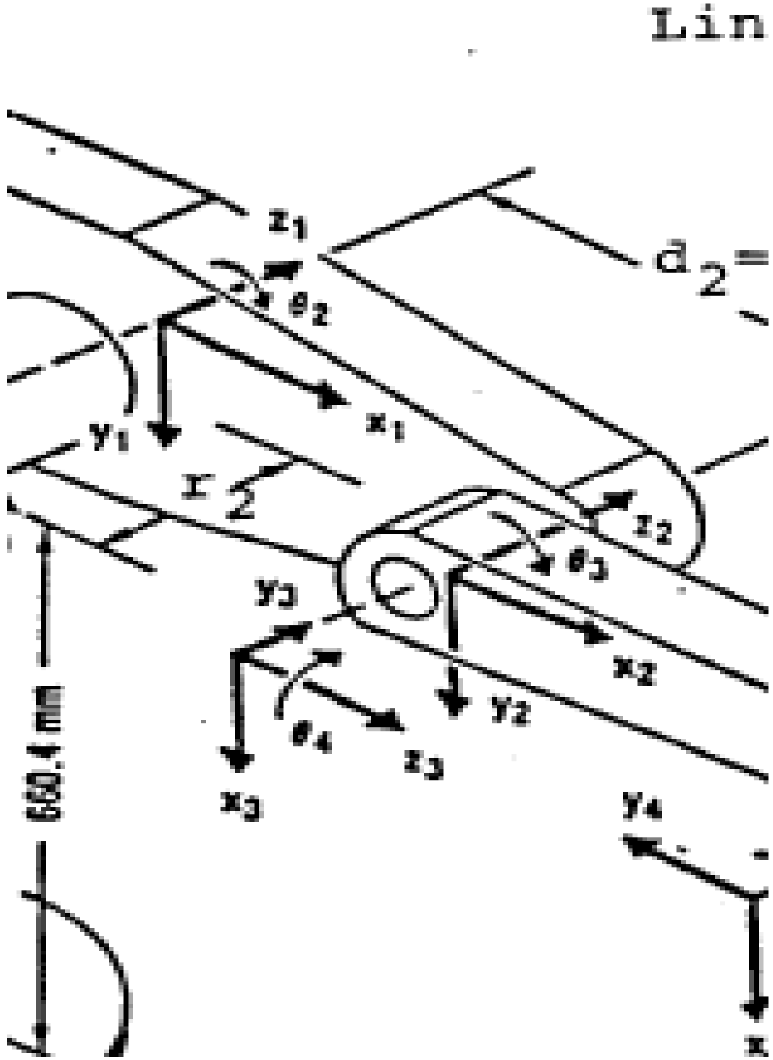
Frames 0 to 1



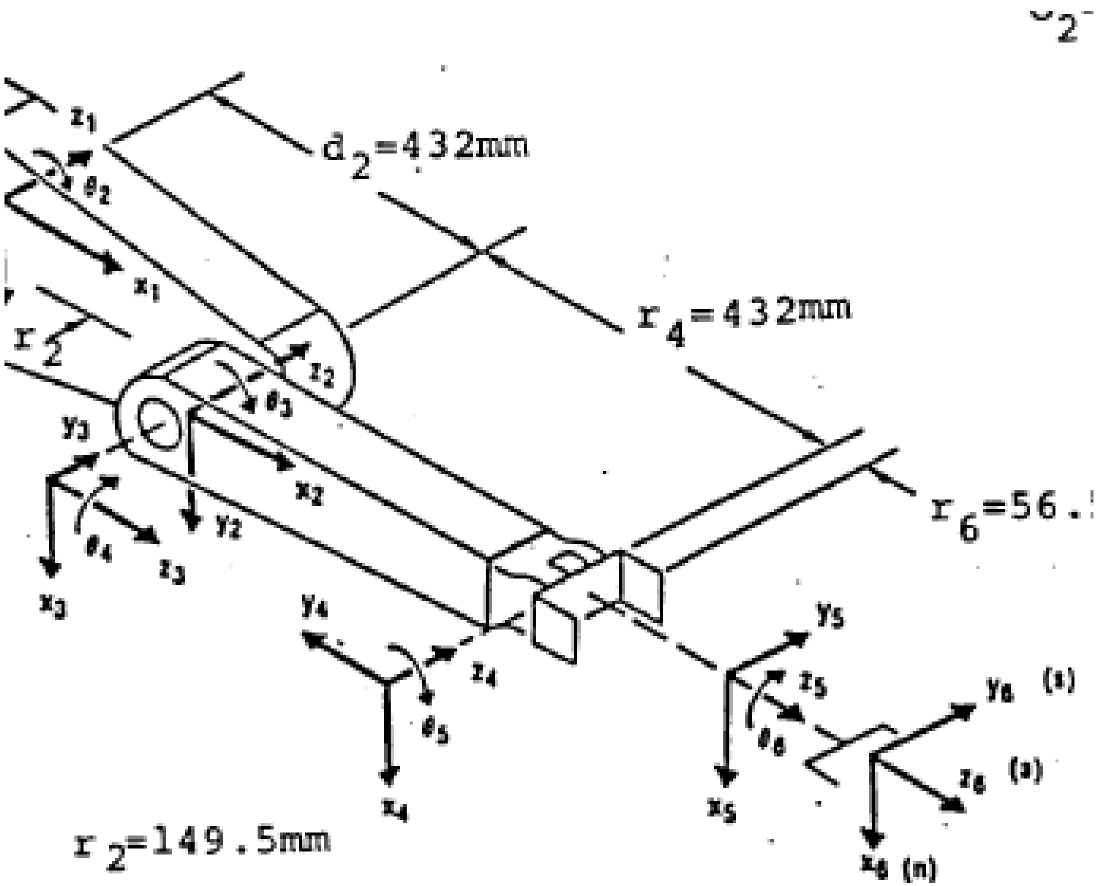
Frames 1 to 2


Lin
z_1
d_2=
z_0
θ_2
θ_1
x_1
y_1
x_0
r_2
z_2
y_0
θ_3
y_3
x_2
y_2
660.4 mm
θ_4
z_3
x_3
y_4
x

Frames 2 to 3


Lin
z_1
θ_2
$d_2=$
y_1
x_1
r_2
z_2
θ_3
y_3
x_2
y_2
θ_4
z_3
x_3
680.4 mm
y_4
x

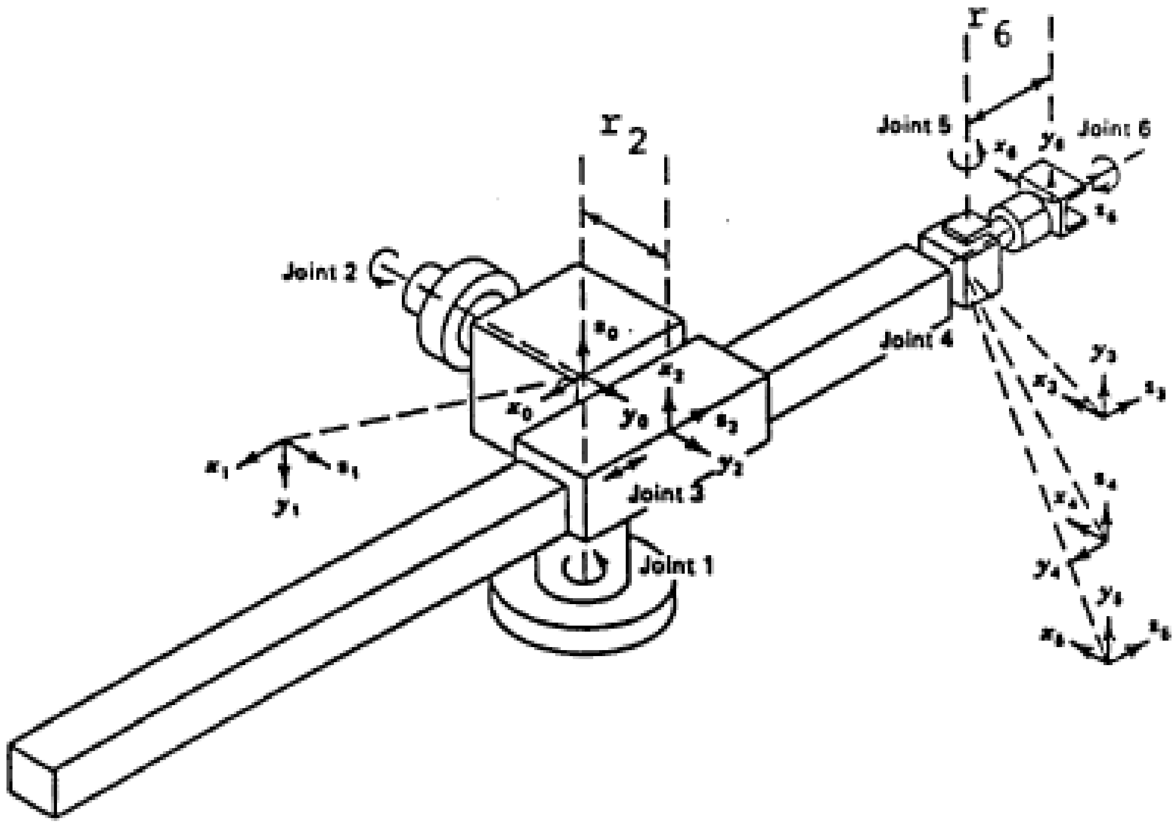
Frames 3 to 4, to 5, to 6



Robot Kinematic Modeling

EXAMPLE 2: The Stanford Arm

Link Coordinate Systems:

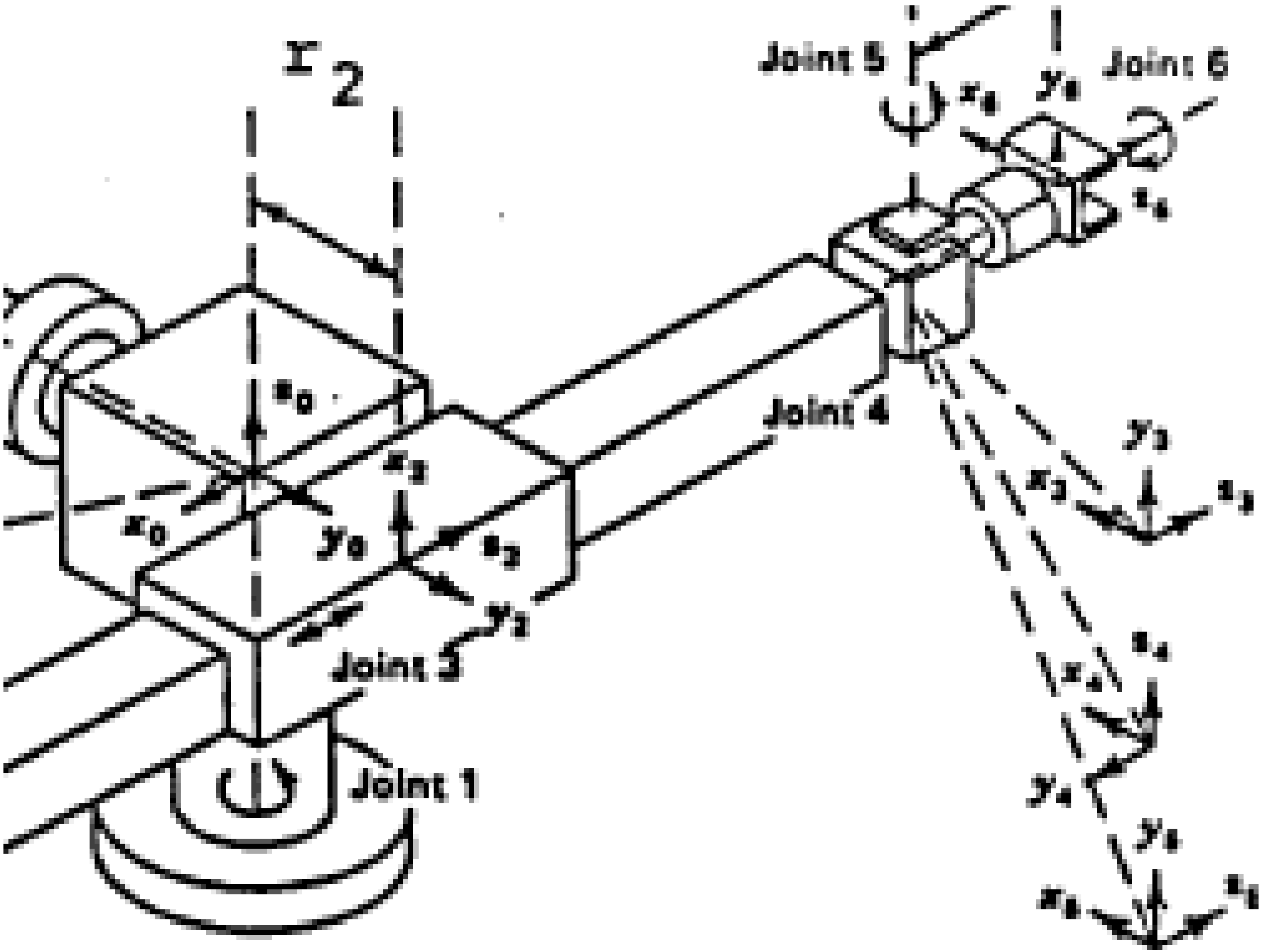


$O_1 = O_0$

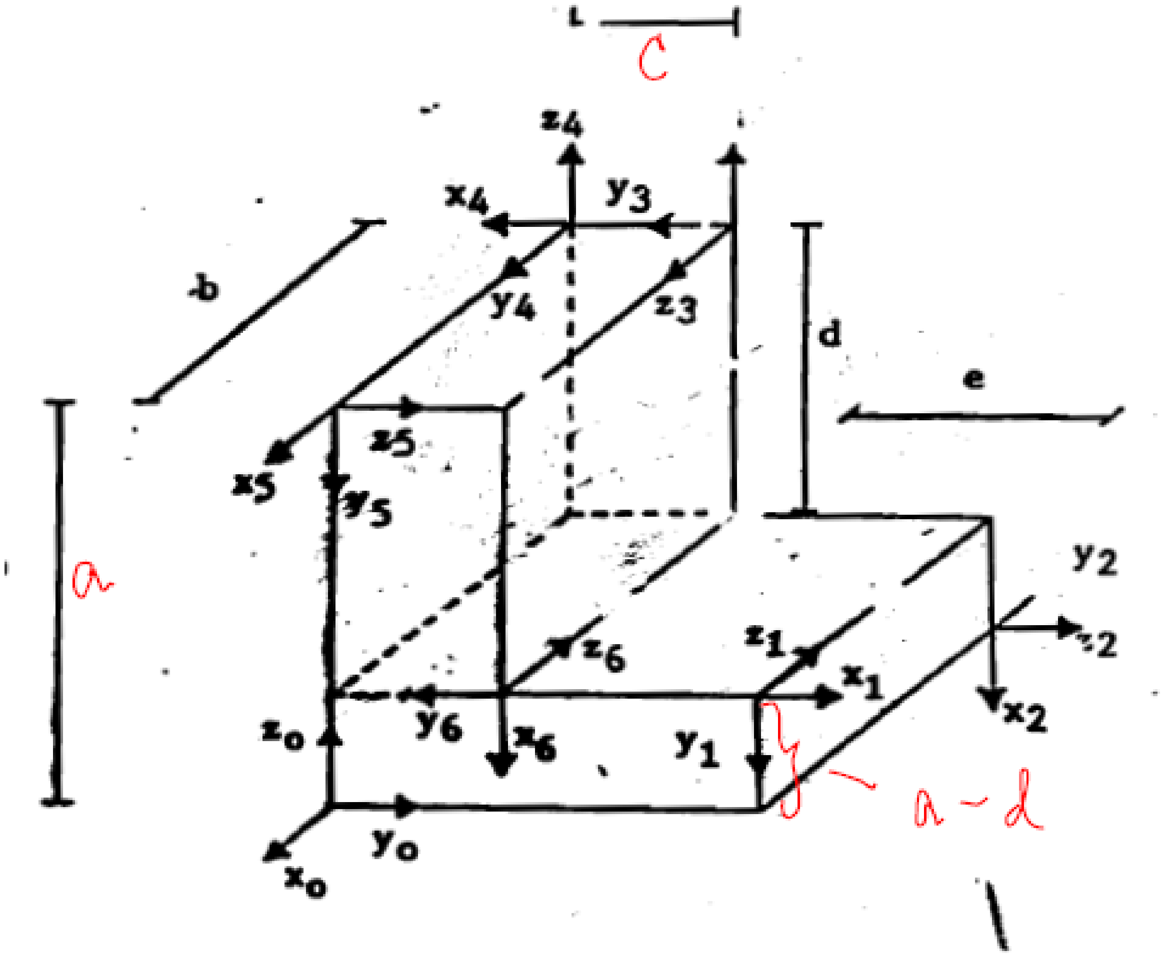
$O_3 = O_4 = O_5$

$r_2 = 16.2\text{cm}$

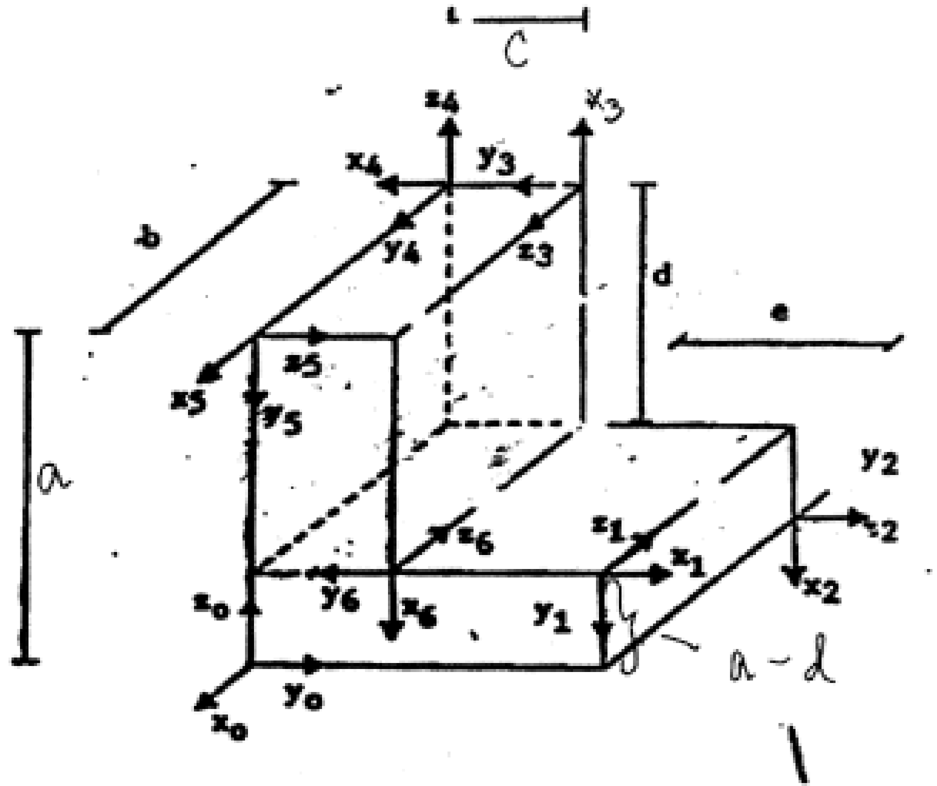
$r_6 = 24.7\text{cm}$



Joint 5
Joint 6
Joint 4
Joint 3
Joint 1

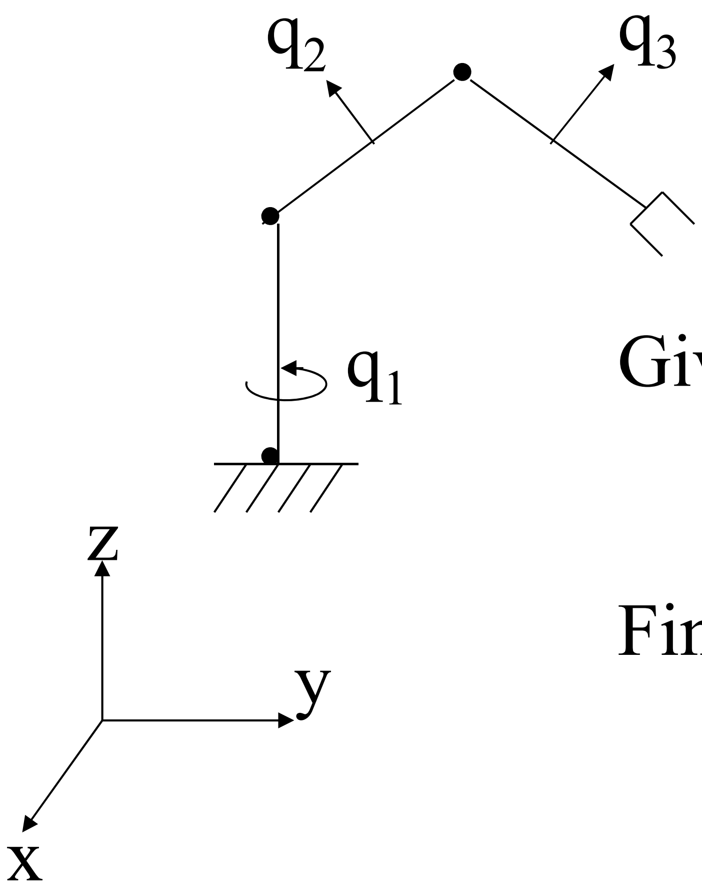


i	θ_i	r_i	d_i	α_i
1				
2				
3				
4				
5				
6				



i	θ_i	r_i	d_i	α_i
1	+90°	a-d	c+e	-90°
2	+90°	b	a-d	90°
3	180°	-e	a	-90°
4	+90°	0	c	+90°
5	+90°	0	b	-90°
6	+90°	c	d	-90°

Forward Kinematic Problem



Given: q1, q2, q3….
(n joint positions)

Find: End-Effector position P_E
and orientation R_E
(m end-effector parameters)

Forward Kinematic Problem

1. Assign Cartesian Coordinate frames to each link (including the base ϕ & end-effector N)

2. Identify the joint variables and link kinematic parameters

3. Define the link transformation matrices. ${}^{i-1}T_i = A_i$

4. Compute the forward transformation

$${}^{0}T_N(q_1, q_2, \ldots, q_N) = A_1 A_2 A_3 \ldots A_N = \begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Inverse Kinematic Problem

Given: Position & Orientation of END-EFFECTOR

Find: joint coordinates

$$\underline{{}^{0}T_{N}} \longrightarrow q_1, q_2, q_3, \ldots, q_N$$

12 equations in N unknowns

Need to solve at most six independent equations in N unknowns.

Inverse Kinematic Problem

<u>ISSUES</u>

- Existence of solutions
  - Workspace
  - Dextrous Workspace
  - Less than 6 joints
  - Joint limits (practical)
- Multiple solutions
  - Criteria
  - Solvability
    - closed form
      - Algebraic
      - Geometric
    - numerical
  - number of solutions

    = 16 $\underline{d_i}, \underline{r_i} \neq 0$ <u>for six points</u>

Solution To Inverse Kinematics

$$^0T_N = {}^0T_1\,{}^1T_2\,{}^2T_3 \ldots {}^{N-1}T_N = A_1 A_2 A_3 \ldots A_N$$

Given: $^0T_N = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ \phi & \phi & \phi & 1 \end{bmatrix}$ $\quad$ $Ai = \begin{bmatrix} c\theta_i & -c\alpha_i s\theta_i & s\alpha_i s\theta_i & d_i c\theta_i \\ s\theta_i & c\alpha_i c\theta_i & -s\alpha_i c\theta_i & d_i s\theta_i \\ \phi & s\alpha_i & c\alpha_i & r_i \\ \phi & \phi & \phi & 1 \end{bmatrix}$

Find: $q = q_1, q_2, q_3, \ldots, q_N$ (joint coordinates)

Solution To Inverse Kinematics

$$\begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ \phi & \phi & \phi & 1 \end{bmatrix} = A_1 A_2 A_3 \ldots A_N$$

12 Equations
- 6 independent
- 6 redundant

N unknowns

LHS(i,j) = RHS(i,j)

row
i = 1, 2, 3

column
j = 1, 2, 3, 4

Solution To Inverse Kinematics Problem

General Approach: Isolate one joint variable at a time

$$\underbrace{A_1^{-1}\,{}^0T_N}_{\text{function of } q_1} = A_2A_3\ldots A_N = \underbrace{{}^1T_N}_{\text{function of } q_2,\ \ldots\ ,\ q_N}$$

- Look for constant elements in ${}^1T_N$
- Equate LHS(i,j) = RHS(i,j)
- Solve for q_1

Solution To Inverse Kinematics Problem

$$A_2^{-1} A_1^{-1} {}^0T_N = A_3 \ldots A_N = {}^2T_N$$

${}^2T_N$: function of $q_3, \ldots, q_N$

$A_2^{-1} A_1^{-1} {}^0T_N$: function of q_1, q_2

↳ only one unknown q_2 since q_1 has been solved for

- Look for constant elements of ${}^2T_N$
- Equate LHS(i,j) = RHS(i,j)
- Solve for q_2
- Maybe can find equation involving q_1 only

Note:

- There is no algorithmic approach that is 100% effective
- Geometric intuition is required

Analytic Solution To Inverse Kinematics

There are Two Classes of Robot Geometries for which closed-form inverse kinematic solutions are guaranteed. They are:

1. Robots with any 3 joints TRANSLATIONAL

2. Robots with any 3 rotational joint axes co-intersecting at a common point

These are DECOUPLED ROBOT GEOMETRIES

meaning

- can reduce system to a lower order subsystem (i.e. 3rd-order) for which closed form solutions are guaranteed

General Analytical Inverse Kinematic Formula

Case 1: $\left.\begin{array}{l}\sin\theta = a \\ \cos\theta = b\end{array}\right\}$ $\left.\begin{array}{l}a \in [-1,1] \\ b \in [-1,1]\end{array}\right\}$ $\theta = \text{ATAN2}(a, b)$ unique

Case 2: $\sin\theta = a$ $\quad a \in [-1,1]$ $\quad \theta = \text{ATAN2}(a, \pm\sqrt{1-a^2})$

$\cos\theta = \pm\sqrt{1-a^2}$

2 solutions

$\theta,\ 180^\circ - \theta$

@ $\theta = \pm 90^\circ$, $|a| = 1$, → "boundary" singularity

$\cos\theta = b$ $\quad b \in [-1,1]$ $\quad \theta = \text{ATAN2}(\pm\sqrt{1-b^2}, b)$

$\sin\theta = \pm\sqrt{1-b^2}$

2 solutions

$\theta,\ -\theta$

@ $\theta = 0^\circ, 180^\circ$, $|b| = 1$, → "boundary" singularity

→ degeneracy of order 2

General Analytical Inverse Kinematic Formula

Case 3: $a\cos\theta + b\sin\theta = 0 \longrightarrow \theta = \text{ATAN2}(a, -b)$ or

$\text{ATAN2}(-a, b)$

2 solutions, 180º apart

Singularity when $a = b = 0$

$\rightarrow$ infinite order degeneracy

Case 4: $a\cos\theta + b\sin\theta = c$ $\quad a, b, c \neq 0$ $\quad$ 2 solutions

$$\theta = \text{ATAN2}(b, a) + \text{ATAN2}(\pm\underbrace{\sqrt{a^2 + b^2 - c^2}}, c)$$

≥ 0 For solution to exist

$a^2 + b^2 + c^2 < 0 \rightarrow$ outside workspace

$a^2 + b^2 + c^2 = 0 \rightarrow$ 1 solution (singularity)

degeneracy of order 2

General Analytical Inverse Kinematic Formula

Case 5: $\sin\theta\sin\phi = a$

$\cos\theta\sin\phi = b$

Note that once ϕ is known, θ is unique

$\theta = \text{ATAN2}(a, b)$ if $\sin\phi$ is $\oplus$ positive

$\theta = \text{ATAN2}(-a, -b)$ if $\sin\phi$ is $\ominus$ negative

If $\cos\phi = c \rightarrow \phi = \text{ATAN2}(\pm\sqrt{a^2+b^2}, c)$ (2 solutions for ϕ)

Then 2 solutions:

$\theta = \text{ATAN2}(a, b)$	$\theta = \text{ATAN2}(-a, -b)$
$\phi = \text{ATAN2}(\sqrt{a^2+b^2}, c)$	$\phi = \text{ATAN2}(-\sqrt{a^2+b^2}, c)$

Singularity: $a = b = 0$ $|c| = 1$

θ = undefined ϕ = 1 solution

General Analytical Inverse Kinematic Formula

Case 6: $a\cos\theta - b\sin\theta = c$ (1)

$a\sin\theta + b\cos\theta = d$ (2)

Then $\theta = \text{ATAN2}(ad - bc, ac + bd)$

1 solution

Note that for (1) & (2) to be satisfied, or at (1) & (2), we have

$$a^2 + b^2 = c^2 + d^2$$

Decoupling (Kinematic)

"Finding a subset of joints primarily responsible for the completion of a subset of the manipulator task"

Given a Total Task: Involves the identification of:

– decoupled task (subsets of tasks)
– decoupled robot subsystem responsible for the decoupled task

Decoupled Robot Geometry – refers to a manipulator Geometry for which decoupling is guaranteed

Decoupling (Kinematic)

Decoupled Robot Geometries: (6-axes)

1. Any Three (3) Translational Joints
2. Any Three Co-Intersecting Rotational Axes

} Pieper, 1968

3. Any 2 Transl. Joints Normal to a Rot. Joint
4. Transl. Joint Normal to 2 Parallel Joints
5. Any 3 Rot, Joints Parallel

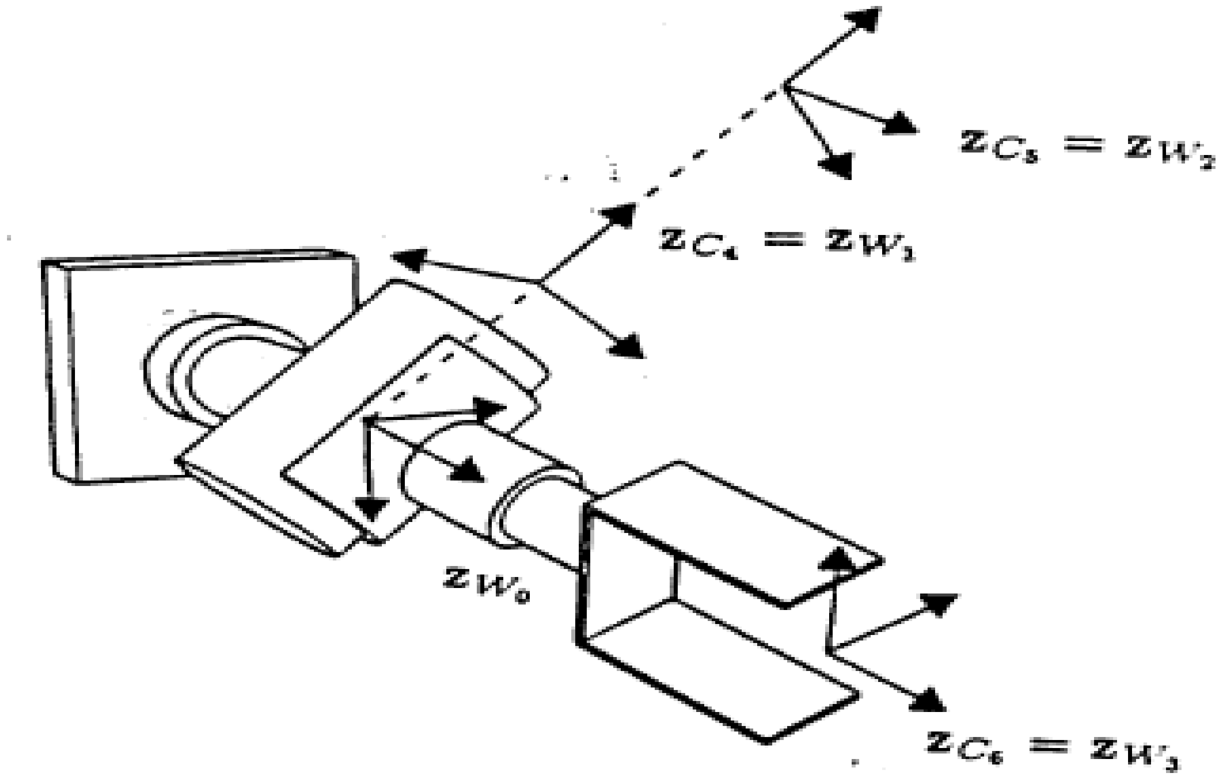
} Ang, 1992

V.D. Tourassis and M.H. Ang Jr., "Task Decoupling in Robot Manipulators," Journal of Intelligent and Robotic Systems 14:283-302, 1995. (Technical Report in 1992).

Decoupling (Kinematic)

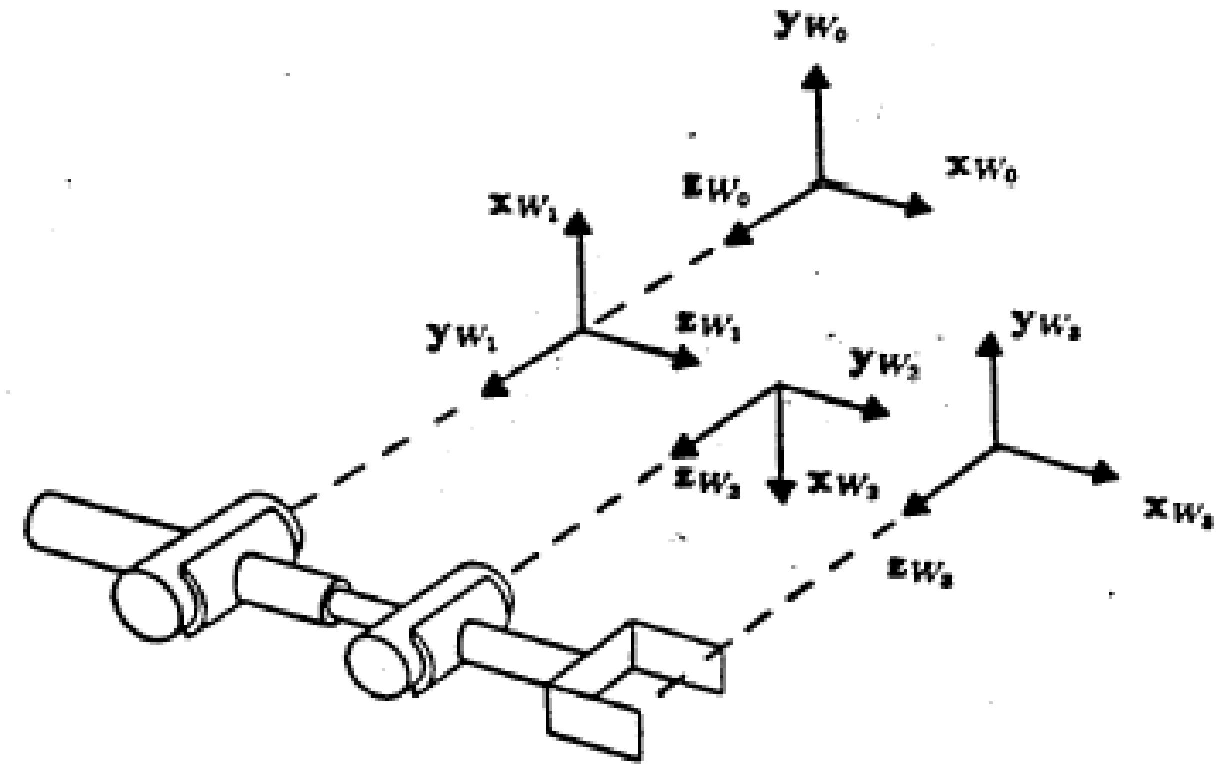
Robots with <u>Spherical Wrists</u> is a popular decoupled robot geometry

→ 3 wrist axes co-intersecting at a common point



Decoupling (Kinematic)

For robots that do not have decoupled geometries, a closed Form solution may not exist, → one has to resort to numerical and iterative procedures.



Numerical Solutions

- m equations in n unknowns (usually $m \leq n$)
- start with an initial estimate for the n unknowns
- compute the error caused by this inaccurate estimate

$^{D}T_N = (T_D)^{-1} T_N$ = position & orientation of end-effector frame with respect to origin of target frame $\rightarrow$ indication of error

$$\Updownarrow$$

$$r_x \; r_y \; r_z \;\; r_\phi \; r_\theta \; r_\varphi$$

- modify estimate to reduce error

Numerical Solutions

Three important requirements for the numerical algorithm are:

i. a priori conditions for convergence
ii. insensitivity to initial estimates
iii. provision for multiple solutions

- The most common methods are based on the Newton-Raphson approach.
  - usually good if initial estimate is close to correct solution (practical?)

Ref: A.A.Goldenberg, B. Benhabib, & R.G.Fenton, "A Complete Generalized Solution to the Inverse Kinematics of Robots" IEEE Journal of Rob. & Auto. 1(1): March 1985, pp. 14-20.