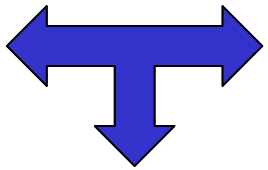
CHAPTER 4

Force Transformation and Robot Statics

Forces Acting on A
Expressed in B



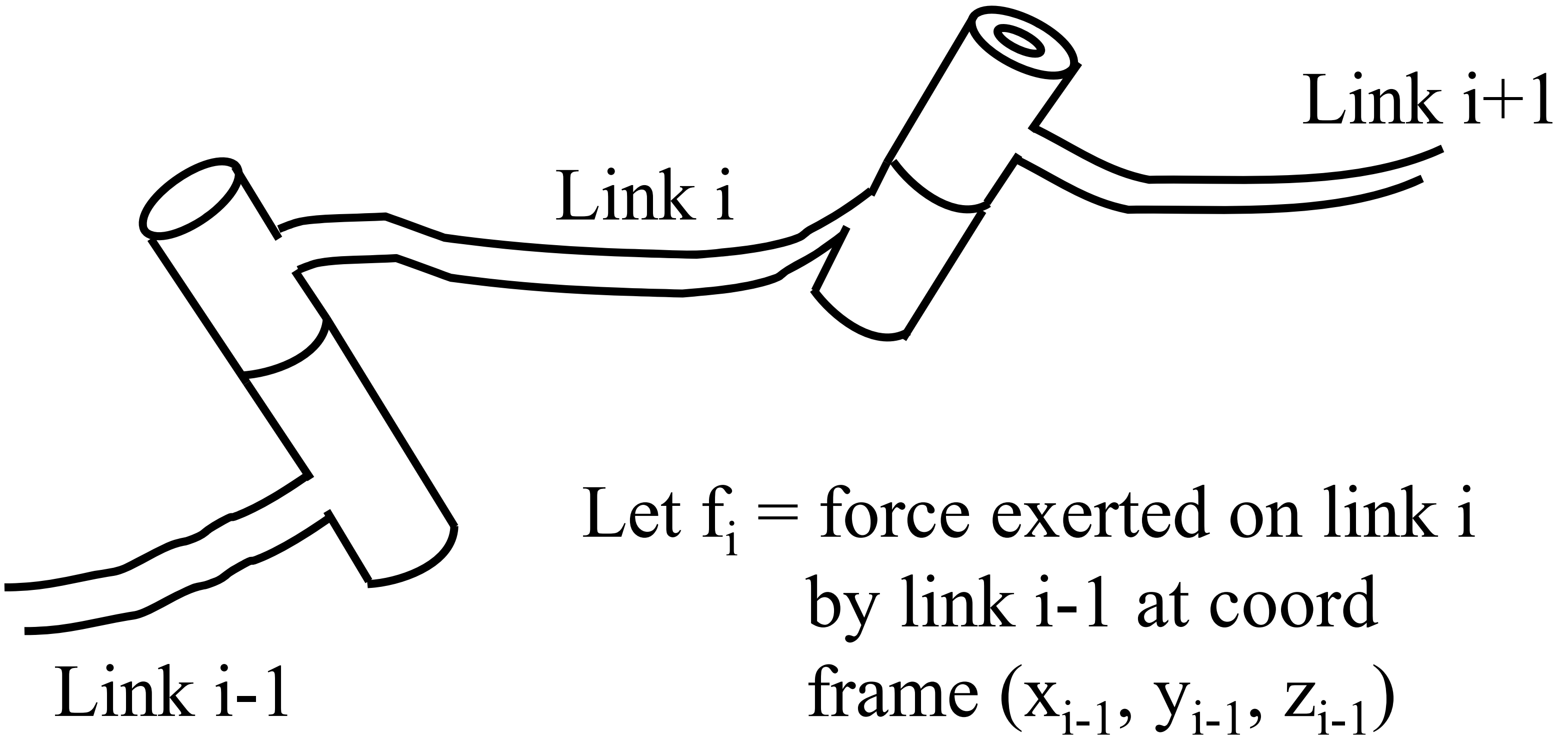
Forces Acting on C
Expressed in D

Actuator Forces in Joints

Learning Objectives

- Relate joint actuator forces with end-effector forces
- Transform forces in different frames
  - Compute forces felt at different parts of links/robots

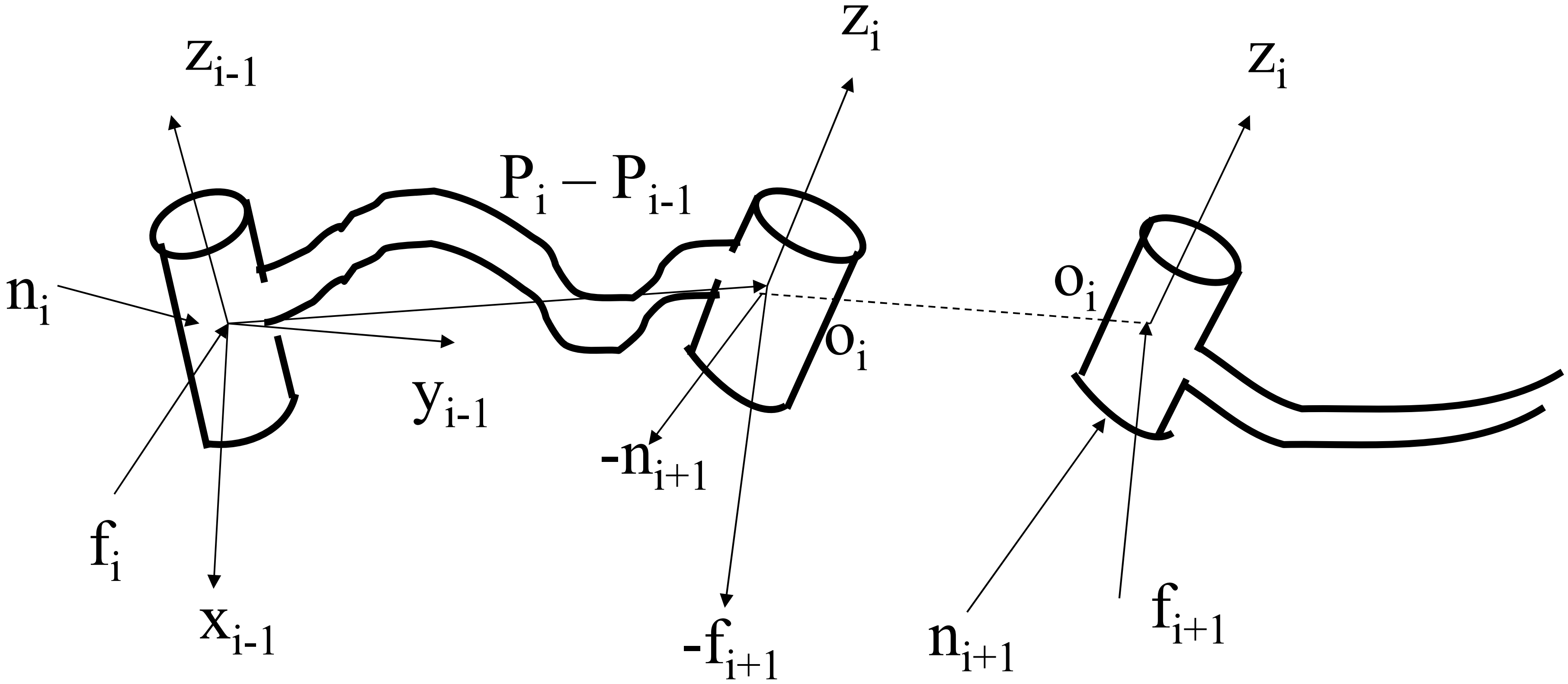
Static Forces in Manipulators



Let f_i = force exerted on link i by link i-1 at coord frame $(x_{i-1}, y_{i-1}, z_{i-1})$

n_i = moment exerted on link i

Static Forces in Manipulators



Static Forces in Manipulators

$\sum \mathbf{F} = 0 \quad \mathbf{f_i} - \mathbf{f_{i+1}} = 0$

$\sum$Torques about origin of frame i-1 = 0

$\mathbf{n_i} - \mathbf{n_{i+1}} + (\mathbf{p_i} - \mathbf{p_{i-1}}) \times (-\mathbf{f_{i+1}}) = 0$

If we start with a description of the force and moment applied by the last link (end-effector) to the environment, we can calculate the force and moment applied by each link working from the last
link down to the base, link ϕ.

$\left.\begin{matrix}\mathbf{f_{n+1}} \\ \mathbf{n_{n+1}}\end{matrix}\right\}$ Force exerted by the manipulator hand on its environment.

Static Forces in Manipulators

Recursive Equations:

$$\left.\begin{aligned} \mathbf{f_i} &= \mathbf{f_{i+1}} \\ \mathbf{n_i} &= \mathbf{n_{i+1}} + (\mathbf{p_i} - \mathbf{p_{i-1}}) \times \mathbf{f_{i+1}} \end{aligned}\right\}$$

all vectors expressed in same frame (e.g. base frame ϕ)

What forces are Needed at the Joints in order to Balance the Reaction Forces & Moments acting in the link

$$\mathbf{T_i} = \begin{cases} \mathbf{n_i}^{\mathbf{T}}\, \mathbf{z_{i-1}} & \text{for a rotational link i} \\ \mathbf{f_i}^{\mathbf{T}}\, \mathbf{z_{i-1}} & \text{for a translational link i} \end{cases}$$

Jacobians In Force Domain

- When forces act on a mechanism, work (in the technical sense) is done if the mechanism moves through a displacement

- Principle of VIRTUAL WORK allows us to make certain statements about the static case by defining a VIRTUAL DISPLACEMENT δx that is experienced without passage of time $dt = 0$ (infinitesimal)

Jacobians In Force Domain

- Since work has units of energy, it must be the same measured in any set of generalized coordinates

6x1 virtual joint disp.

$$\mathbf{F} \bullet \delta\mathbf{x} = \boldsymbol{\tau} \bullet \delta\mathbf{q}$$

6x1
Cartesian
Force-Moment
Vector

6x1
Virtual
displ. In
Cartesian space

Torque/Forces at joints
6x1

Jacobians In Force Domain

- But $\quad \delta\mathbf{x} = \mathbf{J}\,\delta\mathbf{q}$

- Therefore $\quad \mathbf{F}^T \underbrace{[\mathbf{J}\,\delta\mathbf{q}]}_{\delta\mathbf{x}} = \underline{\boldsymbol{\tau}^T\,\delta\mathbf{q}}$

$$\mathbf{F}^T\,\mathbf{J} = \boldsymbol{\tau}^T$$

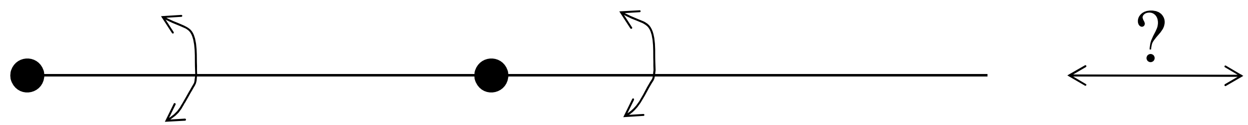
$$\boxed{\boldsymbol{\tau} = \mathbf{J}^T\,\mathbf{F}}$$

expressed in the same (<u>consistent</u>) Frame

Valid only at non-singular configurations

Jacobians In Force Domain

- When the Jacobian loses full rank, there are certain directions in which the end-effector cannot exert static forces (through joint actuation) as desired

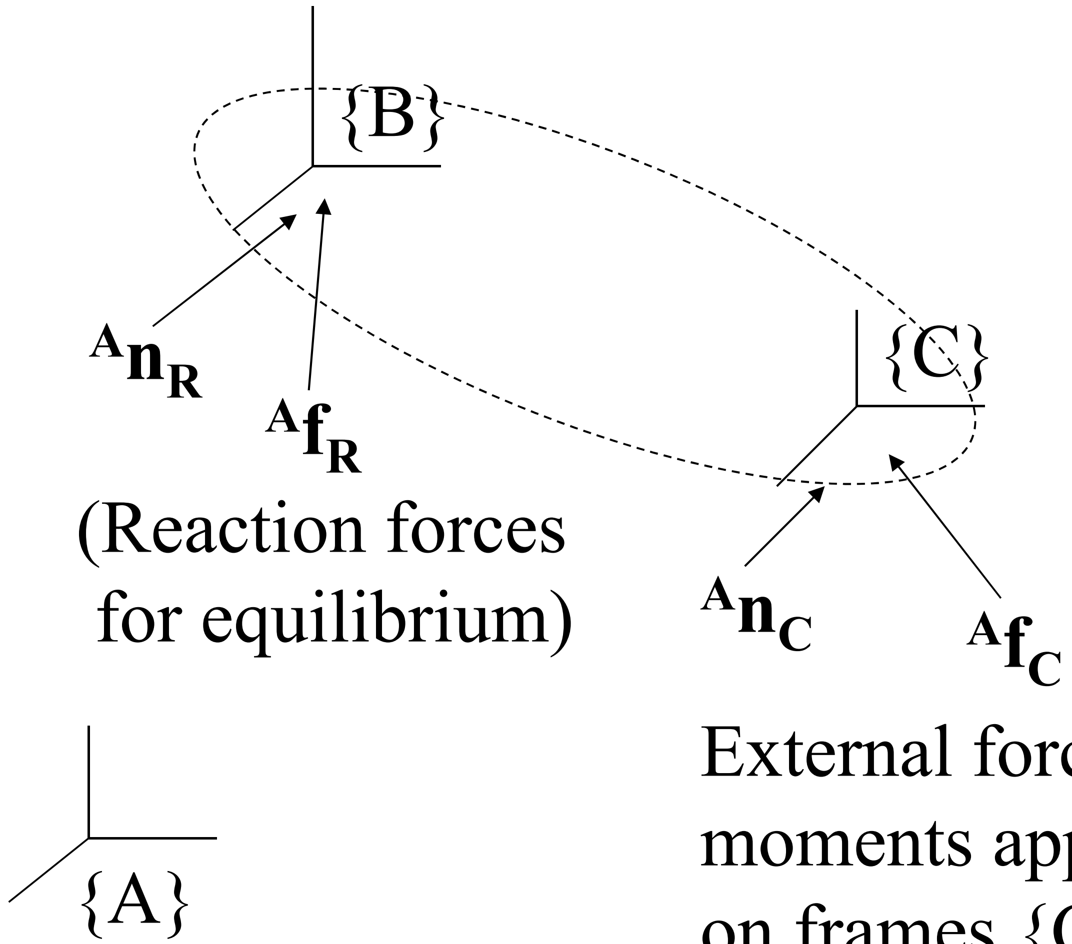


- That is, if **J** is singular, the equation is not valid
  - **F** could be increased or decreased in certain directions with no effect on the value calculated for $\boldsymbol{\tau}$
  - These directions are in the null-space of the Jacobian

Jacobians In Force Domain

- Note that a Cartesian space quantity can be converted into a joint space quantity without calculating any inverse kinematic functions.

Cartesian Transformation Of Static Force



External forces/ moments applied on frames {C}

Given: ${}^{A}f_{C}$

${}^{A}n_{C}$

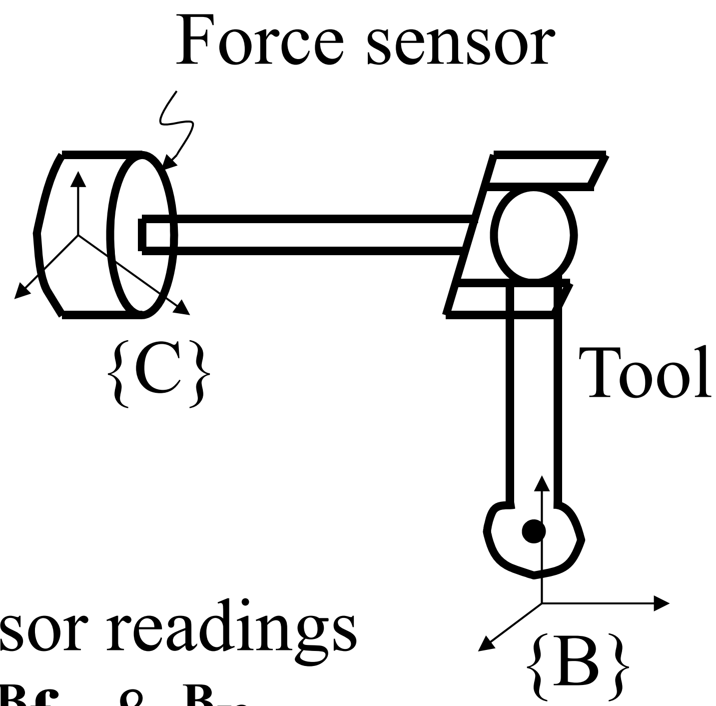
Find: ${}^{A}f_{B}$

${}^{A}n_{B}$

(the force/moment experienced at B if force/moment is exerted on C)

Cartesian Transformation Of Static Force

Why is this important?



$^{C}\mathbf{f}_{C}$ & $^{C}\mathbf{n}_{C}$ can be force sensor readings
But our primary interest is $^{B}\mathbf{f}_{B}$ & $^{B}\mathbf{n}_{B}$
(force/moments at tool tip)

Cartesian Transformation Of Static Force

Equilibrium:

$\sum F = 0 \qquad {}^{A}f_{C} + {}^{A}f_{R} = 0$

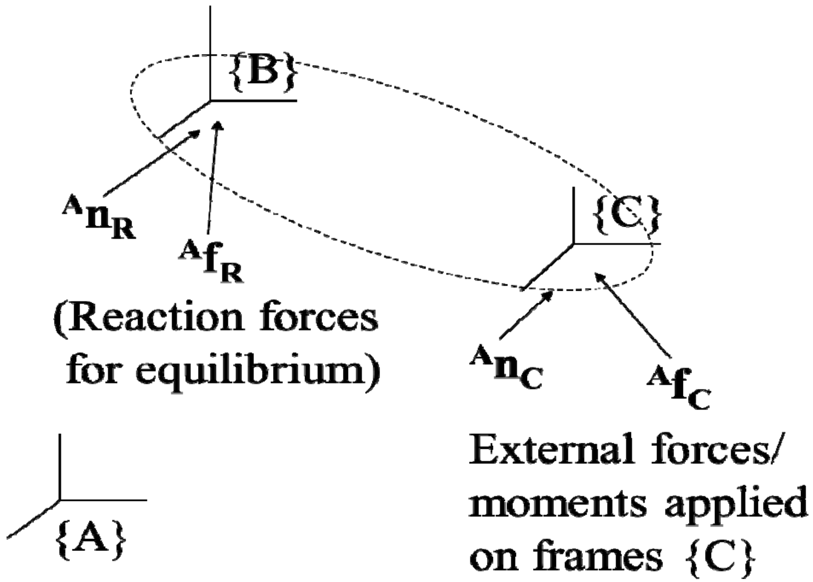
$\qquad\qquad {}^{A}f_{R} = -\,{}^{A}f_{C}$

$\sum N = 0 \qquad {}^{A}n_{C} + ({}^{A}p_{B} - {}^{A}p_{C}) \times {}^{A}f_{R} + n_{R} = 0$

$\qquad\qquad {}^{A}n_{R} = -\,{}^{A}n_{C} - ({}^{A}p_{B} - {}^{A}p_{C}) \times {}^{A}f_{R}$

But ${}^{A}f_{B} = -\,{}^{A}f_{R} \;\longrightarrow$

$$\boxed{{}^{A}f_{B} = {}^{A}f_{C}}$$



Cartesian Transformation Of Static Force

$$ {}^A\mathbf{n}_B = -{}^A\mathbf{n}_R = {}^A\mathbf{n}_C + ({}^A\mathbf{p}_B - {}^A\mathbf{p}_C) \times {}^A\mathbf{f}_R $$

$$ {}^A\mathbf{n}_B = {}^A\mathbf{n}_C + ({}^A\mathbf{p}_B - {}^A\mathbf{p}_C) \times (-{}^A\mathbf{f}_C) $$

$$ \boxed{ {}^A\mathbf{n}_B = {}^A\mathbf{n}_C + ({}^A\mathbf{p}_C - {}^A\mathbf{p}_B) \times {}^A\mathbf{f}_C } $$

OR

$$ {}^A\mathbf{n}_B = {}^A\mathbf{n}_C + (\widehat{{}^A\mathbf{R}_B\, {}^B\mathbf{p}_C}\,)\, {}^A\mathbf{f}_C $$

in Matrix Form

$$ \begin{bmatrix} {}^A f_B \\ {}^A n_B \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} I & 0 \\ \hline \widehat{{}^A R_B\, {}^B p_C} & I \end{array}\right]}_{\text{Force torque Jacobian transformation}} \begin{bmatrix} {}^A f_C \\ {}^A n_C \end{bmatrix} $$

Cartesian Transformation Of Static Force

But in typical applications, we would like to relate

$$\begin{bmatrix} {}^{C}\mathbf{f}_{C} \\ {}^{C}\mathbf{n}_{C} \end{bmatrix} \text{with} \begin{bmatrix} {}^{B}\mathbf{f}_{B} \\ {}^{B}\mathbf{n}_{B} \end{bmatrix}$$

[e.g. sensor readings will be expressed in local frame of sensor]

We can transform vectors **f** & **n** like any other vector via Rotation Matrices

$$\begin{bmatrix} {}^{A}\mathbf{f}_{C} \\ {}^{A}\mathbf{n}_{C} \end{bmatrix} = \left[\begin{array}{c:c} {}^{A}\mathbf{R}_{C} & 0 \\ \hdashline 0 & {}^{A}\mathbf{R}_{C} \end{array}\right] \begin{bmatrix} {}^{C}\mathbf{f}_{C} \\ {}^{C}\mathbf{n}_{C} \end{bmatrix}$$

Cartesian Transformation Of Static Force

$\therefore$

$$
\begin{bmatrix} {}^{A}f_{B} \\ {}^{A}n_{B} \end{bmatrix} = \left[\begin{array}{c:c} I & 0 \\ \hdashline \widehat{{}^{A}R_{B}\,{}^{B}p_{C}} & I \end{array}\right] \left[\begin{array}{c:c} {}^{A}R_{C} & 0 \\ \hdashline 0 & {}^{A}R_{C} \end{array}\right] \begin{bmatrix} {}^{C}f_{C} \\ {}^{C}n_{C} \end{bmatrix}
$$

$$
\begin{bmatrix} {}^{A}f_{B} \\ {}^{A}n_{B} \end{bmatrix} = \left[\begin{array}{c:c} {}^{A}R_{C} & 0 \\ \hdashline \left(\widehat{{}^{A}R_{B}\,{}^{B}p_{C}}\right) {}^{A}R_{C} & {}^{A}R_{C} \end{array}\right] \begin{bmatrix} {}^{C}f_{C} \\ {}^{C}n_{C} \end{bmatrix}
$$

Also

$$
\begin{bmatrix} {}^{\mathbf{B}}\mathbf{f}_{\mathbf{B}} \\ {}^{\mathbf{B}}\mathbf{n}_{\mathbf{B}} \end{bmatrix} = \left[\begin{array}{c:c} {}^{\mathbf{B}}\mathbf{R}_{\mathbf{A}} & 0 \\ \hdashline 0 & {}^{\mathbf{B}}\mathbf{R}_{\mathbf{A}} \end{array}\right] \begin{bmatrix} {}^{\mathbf{A}}\mathbf{f}_{\mathbf{B}} \\ {}^{\mathbf{A}}\mathbf{n}_{\mathbf{B}} \end{bmatrix}
$$

Cartesian Transformation Of Static Force

Therefore

$$
\begin{bmatrix} {}^{B}f_{B} \\ {}^{B}n_{B} \end{bmatrix} = \begin{bmatrix} {}^{B}R_{A} & 0 \\ 0 & {}^{B}R_{A} \end{bmatrix} \begin{bmatrix} {}^{A}R_{C} & 0 \\ \widehat{\left({}^{A}R_{B}\,{}^{B}p_{C}\right)}\,{}^{A}R_{C} & {}^{A}R_{C} \end{bmatrix} \begin{bmatrix} {}^{C}f_{C} \\ {}^{C}n_{C} \end{bmatrix}
$$

$$
= \begin{bmatrix} {}^{B}R_{A}\,{}^{A}R_{C} & 0 \\ \underbrace{{}^{B}R_{A}\,\widehat{\left({}^{A}R_{B}\,{}^{B}p_{C}\right)}\,{}^{A}R_{C}} & {}^{B}R_{A}\,{}^{A}R_{C} \end{bmatrix} \begin{bmatrix} {}^{C}f_{C} \\ {}^{C}n_{C} \end{bmatrix}
$$

$$
\left({}^{B}\mathbf{R}_{A}\,\widehat{\left({}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{p}_{C}\right)}\,{}^{A}\mathbf{R}_{C}\right){}^{C}\mathbf{f}_{C} = {}^{B}\mathbf{R}_{A}\left[\left({}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{p}_{C}\right) \times \left({}^{A}\mathbf{R}_{C}\,{}^{C}\mathbf{f}_{C}\right)\right]
$$

$$
= \left({}^{B}\mathbf{R}_{A}\,{}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{p}_{C}\right) \times \left({}^{B}\mathbf{R}_{A}\,{}^{A}\mathbf{R}_{C}\,{}^{C}\mathbf{f}_{C}\right)
$$

$$
= {}^{B}\mathbf{p}_{C} \times \left({}^{B}\mathbf{R}_{C}\,{}^{C}\mathbf{f}_{C}\right)
$$

$$
= \widehat{{}^{B}\mathbf{p}_{C}}\,{}^{B}\mathbf{R}_{C}\,{}^{C}\mathbf{f}_{C}
$$

Cartesian Transformation Of Static Force

$$\begin{bmatrix} {}^{B}f_{B} \\ {}^{B}n_{B} \end{bmatrix} = \left[\begin{array}{c:c} {}^{B}R_{C} & 0 \\ \hdashline {}^{B}\hat{p}_{C}\,{}^{B}R_{C} & {}^{B}R_{C} \end{array}\right] \begin{bmatrix} {}^{C}f_{C} \\ {}^{C}n_{C} \end{bmatrix}$$

This is the form given in Craig's Book