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ME4245EE4304 **Robotics**

SOLUTIONS - Drill Problem Set 2

1. We observe that the first three joint axes always intersect at the common point $O_0 = O_1 = O_2$. A closed-form inverse kinematic solution is therefore guaranteed. We use the decoupling principle to arrive at the solution. Since the location of the co-intersection point $O_0 = O_1 = O_2$ as seen from the end-effector frame of reference (frame 6) does not change with motion of the first three joints, the position of $O_0 = O_1 = O_2$ expressed in the end-effector frame (frame 6), ${}^6\mathbf{p}_0 = {}^6\mathbf{p}_1 = {}^6\mathbf{p}_2$, is a function of the last three joint coordinates (joints 4, 5, and 6) only. This position ${}^6\mathbf{p}_0$ is directly obtained from the task description ${}^0\mathbf{T}_6$ using:

$$ {}^6\mathbf{p}_0 = -{}^0\mathbf{R}_6^T \, {}^0\mathbf{p}_6, \tag{1} $$

i.e., ${}^6\mathbf{p}_0$ is the last column of the inverse of ${}^0\mathbf{T}_6$. We express ${}^6\mathbf{p}_0$ in terms of the last three joint coordinates by taking the fourth column of ${}^6\mathbf{T}_2 = \mathbf{A}_6^{-1}\mathbf{A}_5^{-1}\mathbf{A}_4^{-1}\mathbf{A}_3^{-1}$:

$$ {}^6\mathbf{p}_0 = {}^6\mathbf{p}_2 = \begin{pmatrix} p_x^* \\ p_y^* \\ p_z^* \end{pmatrix} = \begin{pmatrix} -c_6(L_1c_{45} + L_2c_5) \\ s_6(L_1c_{45} + L_2c_5) \\ L_1s_{45} + L_2s_5 - L_3 \end{pmatrix} \tag{2} $$

The docoupled system consists of the decoupled task (Equation (2)) and decoupled set of joint coordinates 4, 5 and 6 (Equation (1)).

The nonlinear system (2) represents a low-order system of three equations in three unknowns $(\theta_4, \theta_5, \theta_6)$, for which a closed-form solution is guaranteed:

$$ \theta_4 = ATAN2\left(\frac{I_4\sqrt{4L_1^2L_2^2 - ({p_x^*}^2 + {p_y^*}^2 + (p_z^* + L_3)^2 - L_1^2 - L_2^2)^2}}{{p_x^*}^2 + {p_y^*}^2 + (p_z^* + L_3)^2 - L_1^2 - L_2^2}\right) \tag{3} $$

where $4L_1^2L_2^2 - ({p_x^*}^2 + {p_y^*}^2 + (p_z^* + L_3)^2 - L_1^2 - L_2^2)^2$ must be ≥ 0; otherwise, the end-effector position is unreachable.

$$ \theta_6 = ATAN2\left(\frac{I_6 p_y^*}{-I_6 p_x^*}\right) \tag{4} $$

$$ \theta_5 = ATAN2\left(\frac{(L_1c_4 + L_2)(p_z^* + L_3) - L_1s_4(-p_x^*c_6 + p_y^*s_6)}{(L_1c_4 + L_2)(-p_x^*c_6 + p_y^*s_6) + L_1s_4(p_z^* + L_3)}\right) \tag{5} $$

where $I_4 = \pm 1$ in (3) and $I_6 = \pm 1$ in (4). Having solved for $(\theta_4, \theta_5, \theta_6)$, we now compute

$$ {}^3\mathbf{T}_6 = \mathbf{A}_4\mathbf{A}_5\mathbf{A}_6. \tag{6} $$

The orientation of frame 3 in frame 0 is then computed from the the task description:

$$ {}^0\mathbf{T}_3 = {}^0\mathbf{T}_6 \, {}^3\mathbf{T}_6^{-1}. \tag{7} $$

${}^0\mathbf{T}_3$ is a function of the first three joint coordinates only. Having solved for $(\theta_4, \theta_5, \theta_6)$ to satisfy the end-effector position, we know solve for $(\theta_1, \theta_2, \theta_3)$ to satisfy the end-effector orientation by taking the rotation matrix part of ${}^0\mathbf{T}_3 = \mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$ only:

$$ {}^0\mathbf{R}_3 = \begin{pmatrix} N_x & O_x & A_x \\ N_y & O_y & A_y \\ N_z & O_z & A_z \end{pmatrix} = \begin{pmatrix} c_1c_2c_3 + s_1s_3 & -c_1c_2s_3 + s_1c_3 & -c_1s_2 \\ s_1c_2c_3 - c_1s_3 & -s_1c_2s_3 - c_1c_3 & -s_1s_2 \\ -s_2c_3 & s_2s_3 & -c_2 \end{pmatrix} \tag{8} $$

From the A_z elements we have

$$ \theta_2 = ATAN2\left(\frac{I_2\sqrt{1 - A_z^2}}{-A_z}\right), \tag{9} $$

where $I_2 = \pm 1$. From the A_x, A_y, N_z, and O_z elements of (8), and for $A_z^2 \neq 1$, we have:

$$ \theta_1 = ATAN2\left(\frac{-I_2A_y}{-I_2A_x}\right), \tag{10} $$

$$ \theta_3 = ATAN2\left(\frac{I_2O_z}{-I_2N_z}\right). \tag{11} $$

In general, there are eight inverse kinematic solutions corresponding to $I_4 = \pm 1$, $I_6 = \pm 1$, and $I_2 = \pm 1$.

If $A_z = \pm 1$, θ_1 and θ_3 describe the same rotation and cannot be computed separately; one degree-of-freedom is lost.

$$ A_z = -1 \longleftrightarrow \begin{cases} \theta_2 = 0^\circ \\ \theta_1 - \theta_3 = ATAN2\left(\frac{O_x}{N_x}\right) \end{cases} \tag{12} $$

$$ A_z = 1 \longleftrightarrow \begin{cases} \theta_2 = 180^\circ \\ \theta_1 + \theta_3 = ATAN2\left(\frac{O_x}{-N_x}\right) \end{cases} \tag{13} $$

Thus when $A_z = \pm 1$, there are an infinite number of solutions.