National University of Singapore Department of Mechanical Engineering
ME4245 Quiz 2 20 Sept 2006, 16:00-17:10

1. Fig. 1 shows a planar robot with 1st and 3rd joint rotational and the 2nd joint translational. With its 3 joints, the robot is able to position its end effector (E) at a location (x, y) and orient its end-effector at an angle β with respect to the positive x axis.

Cos[q1]-q2 Sin[q1]-Sin[q1+q3]

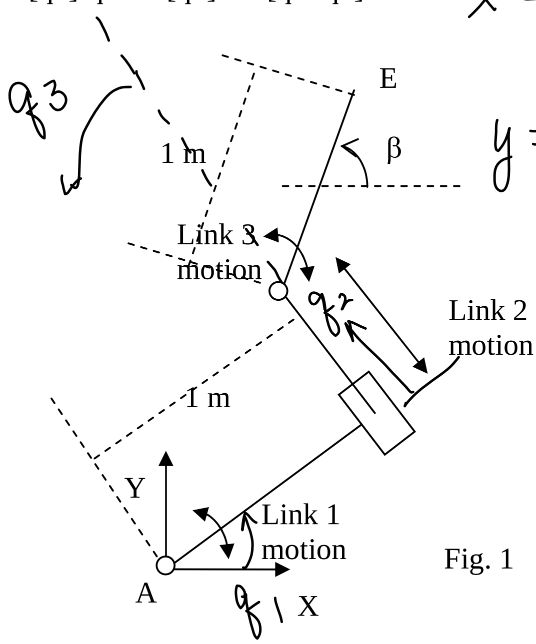


Fig. 1

$$x = c_1 + q_2 \cos(q_1 + 90^\circ) + \cos(q_1 + 90^\circ + q_3)$$
$$= C_1 - q_2 S_1 - S_{13}$$
$$y = s_1 + q_2 \sin(q_1 + 90^\circ) + \sin(q_1 + 90^\circ + q_3)$$
$$= S_1 + q_2 C_1 + C_{13}$$
$$\dot{x} = \frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2 + \frac{\partial x}{\partial q_3}\dot{q}_3$$
$$\dot{y} = \frac{\partial y}{\partial q_1}\dot{q}_1 + \frac{\partial y}{\partial q_2}\dot{q}_2 + \frac{\partial y}{\partial q_3}\dot{q}_3$$
$$\omega_z = \dot{q}_1 + \dot{q}_3$$

a. Derive the complete manipulator Jacobian for this robot.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} u \\ \omega \end{pmatrix} = \underbrace{\begin{pmatrix} -q_2 C_1 - C_{13} - S_1 & -S_1 & -C_{13} \\ -q_2 S_1 - S_{13} + C_1 & C_1 & -S_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{\text{Jacobian, } J \ (6 \times 3)} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix}$$

b. If the task of the robot is (x, y, β), are there singularities? If so, indicate the singular configuration(s), i.e., describe the robot joint coordinates in which the robot is singular.

By inspection, singular configuration is when q2 = 0.

This can also be verified mathematically:

We take only 1st 2 rows and last row of the Jacobian (corresponding to the task).

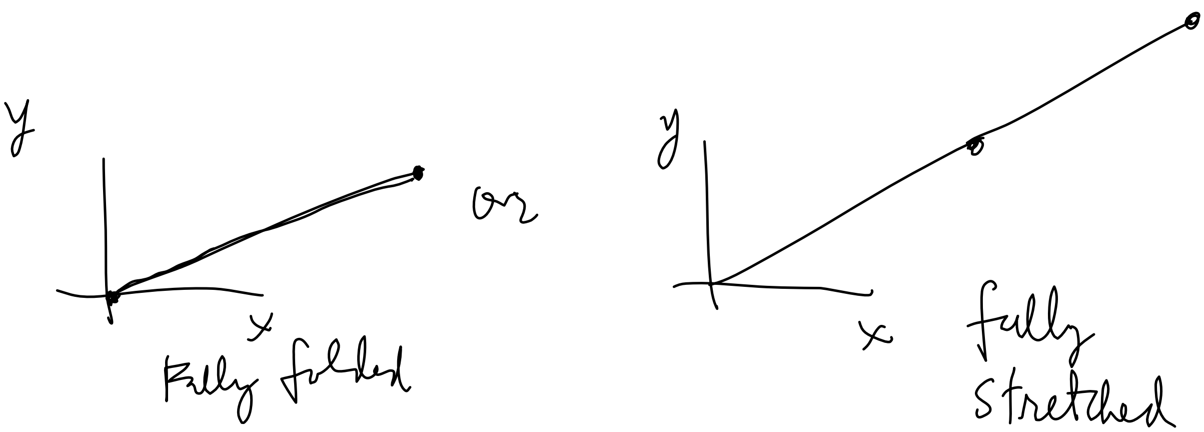
$$J = \begin{pmatrix} -q_2 C_1 - C_{13} - S_1 & -S_1 & -C_{13} \\ -q_2 S_1 - S_{13} + C_1 & C_1 & -S_{13} \\ 1 & 0 & 1 \end{pmatrix}$$

$$\det(J) = q_2$$

$$\det(J) = 0 \rightarrow q_2 = 0 \quad //$$

c. If the task of the robot is (x, y), are there singularities? If so, indicate the singular configuration(s), i.e., describe the robot joint coordinates in which the robot is singular.

By inspection, singular configuration is when (Cos[q3]=0 and q2 = 0)



We can also verify or derive this mathematically. For this task, we take only the 1st two rows of the Jacobian

$$\underset{2\times 3}{J} = \begin{pmatrix} -q_2 C_1 - C_{13} - S_1 & -S_1 & -C_{13} \\ -q_2 S_1 - S_{13} + C_1 & C_1 & -S_{13} \end{pmatrix}$$

Singularities occur when Rank of J becomes less than 2.

When rank is less then 2, 2 joints are not able to provide two degrees of freedom for the task.

So we need to take all possible subsets of 2 joints to find singularities.Take all possible 2x 2 matrices, singularities occur at configurations where the determinant of these matrices are **all** zero.

Joints 1 & 2 (Cols 1 & 2)

$$\det(J)_{2\times 2} = -q_2 - C_3$$

Joints 1 & 3 (Cols 1 & 3)

$$\det(J_{2\times 2}) = C_3 + q_2 S_3$$

Joints 2 & 3 (Cols 2 & 3)

$$\det(J_{2\times 2}) = C_3$$

Singularities occur at joint coordinates where the following 3 equations are satisfied

$$\left.\begin{array}{l} -q_2 - C_3 = 0 \\ C_3 + q_2 S_3 = 0 \\ C_3 = 0 \end{array}\right\} \quad \boxed{q_2 = C_3 = 0}$$

or

$$q_2 = 0, \quad \theta_3 = \begin{cases} 90^\circ \\ -90^\circ \end{cases}$$

d. If the task of the robot is (x, β), are there singularities? If so, indicate the singular configuration(s), i.e., describe the robot joint coordinates in which the robot is singular.

By inspection, when q1 = 0, or 180 deg **and** q2 = 0

Mathematically, we take only the 1st and last rows of the Jacobian, and evaluate determinants of the three 2x2 matrices:

$$\left.\begin{array}{ll} \text{Joint 1 \& 2} & \det(J) = S_1 = 0 \\ \text{Joints 1, 3} & \det(J) = -q_2 C_1 - S_1 = 0 \\ \text{Joints 2 \& 3} & \det(J) = -S_1 = 0 \end{array}\right\} \quad \begin{array}{c} S_1 = 0 \\ \text{and} \\ q_2 = 0 \end{array}$$

e. If the task of the robot is (y, β), are there singularities? If so, indicate the singular configuration(s), i.e., describe the robot joint coordinates in which the robot is singular.

Again, by inspection, when cos[q1] = 0 **and** q2 = 0 (or [(q1 = 90 or -90) **and** q2 = 0]

Mathematically, we take only the 2nd and last rows of the Jacobian, and evaluate determinants of the following three 2x2 matrices:

Joints 1 & 2 $\quad \det(J) = -C_1 = 0$

Joints 1 & 3 $\quad \det(J) = C_1 - q_2 S_1 = 0$

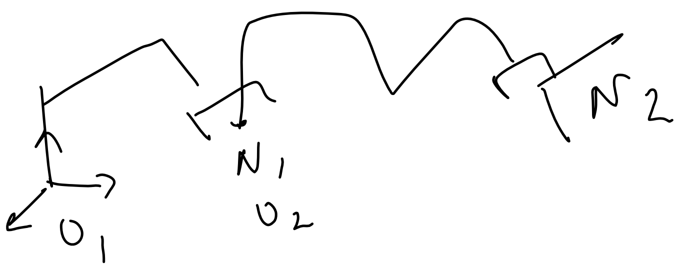
Joints (2 & 3) $\quad \det(J) = C_1 = 0$

$\left.\right\}$ $C_1 = 0$ and $q_2 = 0$

2. Fig 2 shows two robotic manipulators whose kinematic models have been derived. The kinematic models include the derivations of their respective manipulator Jacobians. O_1 and O_2 are the “base frames” of each robot, while N_1 and N_2 are the end-effector frames. Robot #1 has N_1 joints while Robot #2 has N_2 joints. Robot #2 is rigidly attached to the end-effector of Robot #1, with ${}^{N_1}T_{O_2}$ known (this 4 x 4 matrix describes the relative position and orientation of the end-effector of Robot #1 and base of Robot #2, this is constant). Derive an expression for the Jacobian of the entire manipulator system containing $N_1 + N_2$ joints. This Jacobian is used to compute the 6 x 1 velocity of Frame N_2 with respect to Frame O_1 given the N_1+N_2 joint velocities.

$v_1 = J_1 \dot{q}_1$ $\qquad\qquad$ $V_2 = J_2 \dot{q}_2$

Fig 2



$${}^{O_1}u_{N_2} = {}^{O_1}u_{N_1} + {}^{O_1}R_{O_2}\,{}^{O_2}u_{N_2} + {}^{O_1}\omega_{N_1} \times \left({}^{O_1}R_{N_1}\,{}^{N_1}p_{N_2}\right)$$

$$= \quad '' \quad\quad '' \quad - {}^{O_1}R_{N_1}\,{}^{N_1}p_{N_2} \times {}^{O_1}\omega_{N_1}$$

$$= \quad '' \quad\quad '' \quad - {}^{O_1}R_{N_1}\,\widehat{{}^{N_1}p_{N_2}}\,{}^{O_1}\omega_{N_1}$$

$${}^{O_1}\omega_{N_2} = {}^{O_1}\omega_{N_1} + {}^{O_1}R_{O_2}\,{}^{O_2}\omega_{N_2}$$

$J_2\,\dot{q}_2$

Combining:

$$\underbrace{\begin{pmatrix} {}^{O_1}u_{N_2} \\ {}^{O_1}\omega_{N_2} \end{pmatrix}}_{J\,\dot{q}} = \underbrace{\left(\begin{array}{c|c} I & -{}^{O_1}R_{N_1}\,\widehat{{}^{N_1}p_{N_2}} \\ \hline 0 & I \end{array}\right)}_{A} \underbrace{\begin{pmatrix} {}^{O_1}u_{N_1} \\ {}^{O_1}\omega_{N_1} \end{pmatrix}}_{J_1\,\dot{q}_1} + \underbrace{\left(\begin{array}{c|c} {}^{O_1}R_{O_2} & 0 \\ \hline 0 & {}^{O_1}R_{O_2} \end{array}\right)}_{B} \underbrace{\begin{pmatrix} {}^{O_2}u_{N_2} \\ {}^{O_2}\omega_{N_2} \end{pmatrix}}_{J_2\,\dot{q}_2}$$

$$J = \begin{bmatrix} A J_1 & B J_2 \end{bmatrix}$$

$(N_1 + N_2)$