CHAPTER 1

Position and Orientation Definitions and Transformations

Learning Objectives

- Describe position and orientation of rigid bodies relative to each other
- Mathematically represent relative position and orientation of rigid bodies
- Be able to physically visualize these mathematical representations

SPATIAL DESCRIPTION

- used to specify spatial attributes of various objects with which a manipulation system deals

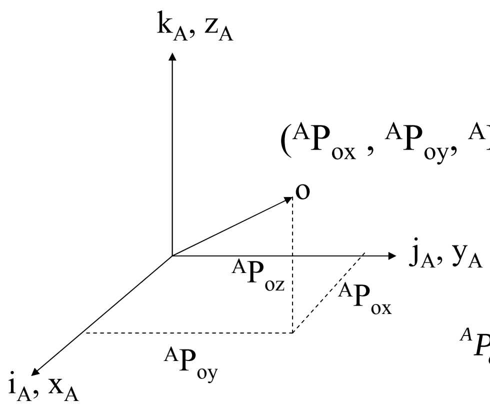
- universe coordinate frame is implicit

- use Cartesian coordinate frames (x, y and z axes = we will use i, j, k, respectively)

POSITION

- attribute of a point

$$^{A}P_{o} = \begin{bmatrix} ^{A}P_{ox} \\ ^{A}P_{oy} \\ ^{A}P_{oz} \end{bmatrix} \in \Re^{3}$$



$(^{A}P_{ox}, {}^{A}P_{oy}, {}^{A}P_{oz})$ = Cartesian coordinates of pt.0 expressed in frame A

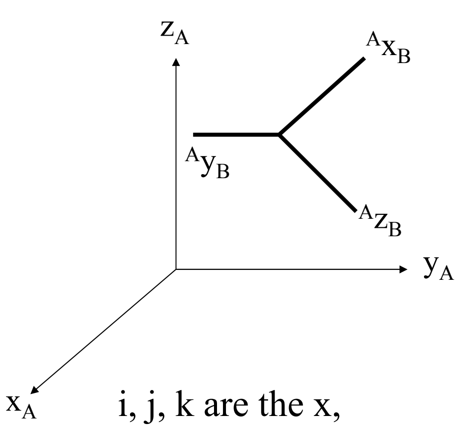
$$^{A}P_{o} = {}^{A}P_{o,x}\,{}^{A}\vec{i} + P_{o,y}\,{}^{A}\vec{j} + P_{o,z}\,{}^{A}\vec{k}$$

Position vector - (coords has magnitude & direction, origin impt)

ORIENTATION

- attribute of a body, relative to some reference frame
- attach a Cartesian coordinate frame to the body (i,j,k axes)
- orientation of each axis of the Cartesian coordinate frame
  - free vectors = orientation vectors = describe orientation
    - magnitude & origin not important
    - only direction is important

ORIENTATION



i, j, k are the x, y and z axes

$({}^A i_B, {}^A j_B, {}^A k_B)$

- unit vectors along each axis of frame B
- free vectors, only direction is relevant, expressed in A

$$
{}^A R_B = \left({}^A i_B \;\; {}^A j_B \;\; {}^A k_B \right)
= \begin{pmatrix} {}^A i_{Bx} & {}^A j_{Bx} & {}^A k_{Bx} \\ {}^A i_{By} & {}^A j_{By} & {}^A k_{By} \\ {}^A i_{Bz} & {}^A j_{Bz} & {}^A k_{Bz} \end{pmatrix} \in \Re^{3\times 3}
$$

Rotation Matrix

- Since each column of the rotation matrix represents unit vectors along the x, y and z directions of the Cartesian coordinates, then they should be orthogonal to each other
- Each column obeys unit length constraints

- Cartesian coord frame : right hand rule
  $\therefore \det({}^{A}R_{B}) = +1$

$\to$ ${}^{A}R_{B}$ is a Proper Orthogonal Matrix
${}^{A}R_{B}^{-1} = {}^{A}R_{B}^{T} = {}^{B}R_{A}$ or ${}^{A}R_{B}\,{}^{A}R_{B}^{T} = I$

Rotation Matrix

- All possible orientations of a rigid body (i.e. coordinate frame attached to the body) can be uniquely specified by a rotation matrix

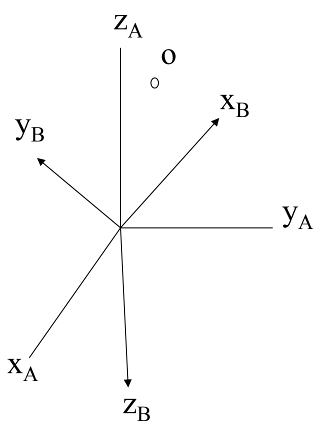
- Orientation has 3 degrees-of-freedom in 3D space, i.e., 3 independent parameters are only needed

Examples

Learning Objectives

- Transformation of position and orientation represents
  - Coordinates Expressed in Frame A → Transformed to be expressed in Frame B
- New position and orientation after motion (i.e., rigid body motion)
- Duality between **transformations** and **rigid body motions**

Coordinate Transformations & Rigid Body Motion



$$^{A}P_o = {}^{A}P_{o,x}\ {}^{A}\overrightarrow{i_A} + P_{o,y}\ {}^{A}\overrightarrow{j_A} + P_{o,z}\ {}^{A}\overrightarrow{k_A} =$$

$$\begin{pmatrix} {}^{A}i_{A,x} & {}^{A}j_{A,x} & {}^{A}k_{A,x} \\ {}^{A}i_{A.y} & {}^{A}j_{A,y} & {}^{A}k_{A,y} \\ {}^{A}i_{A,z} & {}^{A}j_{A,z} & {}^{A}k_{A,z} \end{pmatrix} \begin{pmatrix} {}^{A}P_{o,x} \\ {}^{A}P_{o,y} \\ {}^{A}P_{o,z} \end{pmatrix}$$

$$^{B}P_o = {}^{A}P_{o,x}\ {}^{B}\overrightarrow{i_A} + {}^{A}P_{o,y}\ {}^{B}\overrightarrow{j_A} + {}^{A}P_{o,z}\ {}^{B}\overrightarrow{k_A} =$$

$$\begin{pmatrix} {}^{B}i_{A,x} & {}^{B}j_{A,x} & {}^{B}k_{A,x} \\ {}^{B}i_{A,y} & {}^{B}j_{A,y} & {}^{B}k_{A,y} \\ {}^{B}i_{A,z} & {}^{B}j_{A,z} & {}^{B}k_{A,z} \end{pmatrix} \begin{pmatrix} {}^{A}P_{o,x} \\ {}^{A}P_{o,y} \\ {}^{A}P_{o,z} \end{pmatrix} = {}^{B}R_A\ {}^{A}P_o$$

Coordinate Transformation

- Given:
  - Position of O in A
  - Orientation of O in A
- Position of O in B is:

$$^{B}P_o = {}^{A}P_{o,x}\ {}^{B}\overrightarrow{i_A} + {}^{A}P_{o,y}\ {}^{B}\overrightarrow{j_A} + {}^{A}P_{o,z}\ {}^{B}\overrightarrow{k_A} =$$

$$\begin{pmatrix} {}^{B}i_{A,x} & {}^{B}j_{A,x} & {}^{B}k_{A,x} \\ {}^{B}i_{A,y} & {}^{B}j_{A,y} & {}^{B}k_{A,y} \\ {}^{B}i_{A,z} & {}^{B}j_{A,z} & {}^{B}k_{A,z} \end{pmatrix} \begin{pmatrix} {}^{A}P_{o,x} \\ {}^{A}P_{o,y} \\ {}^{A}P_{o,z} \end{pmatrix} = {}^{B}R_A\ {}^{A}P_o$$

- How about Orientation of O in B?

$$
{}^{B}i_{o} = {}^{A}i_{o,x}\,{}^{B}\overrightarrow{i_{A}} + {}^{A}i_{o,y}\,{}^{B}\overrightarrow{j_{A}} + {}^{A}i_{o,z}\,{}^{B}\overrightarrow{k_{A}} = \begin{pmatrix} {}^{B}i_{A,x} & {}^{B}j_{A,x} & {}^{B}k_{A,x} \\ {}^{B}i_{A,y} & {}^{B}j_{A,y} & {}^{B}k_{A,y} \\ {}^{B}i_{A,z} & {}^{B}j_{A,z} & {}^{B}k_{A,z} \end{pmatrix} \begin{pmatrix} {}^{A}i_{o,x} \\ {}^{A}i_{o,y} \\ {}^{A}i_{o,z} \end{pmatrix} = {}^{B}R_{A}\;{}^{A}i_{o}
$$

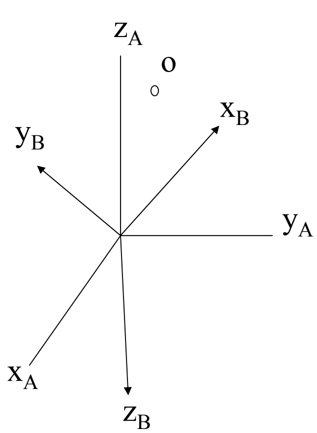
$$
{}^{B}j_{o} = \begin{pmatrix} {}^{B}i_{A,x} & {}^{B}j_{A,x} & {}^{B}k_{A,x} \\ {}^{B}i_{A,y} & {}^{B}j_{A,y} & {}^{B}k_{A,y} \\ {}^{B}i_{A,z} & {}^{B}j_{A,z} & {}^{B}k_{A,z} \end{pmatrix} \begin{pmatrix} {}^{A}j_{o,x} \\ {}^{A}j_{o,y} \\ {}^{A}j_{o,z} \end{pmatrix} = {}^{B}R_{A}\;{}^{A}j_{o}
$$

$$
{}^{B}k_{o} = \begin{pmatrix} {}^{B}i_{A,x} & {}^{B}j_{A,x} & {}^{B}k_{A,x} \\ {}^{B}i_{A,y} & {}^{B}j_{A,y} & {}^{B}k_{A,y} \\ {}^{B}i_{A,z} & {}^{B}j_{A,z} & {}^{B}k_{A,z} \end{pmatrix} \begin{pmatrix} {}^{A}k_{o,x} \\ {}^{A}k_{o,y} \\ {}^{A}k_{o,z} \end{pmatrix} = {}^{B}R_{A}\;{}^{A}k_{o}
$$

$$
\begin{pmatrix} {}^{B}i_{o} & {}^{B}j_{o} & {}^{B}k_{o} \end{pmatrix} = \begin{pmatrix} {}^{B}i_{A,x} & {}^{B}j_{A,x} & {}^{B}k_{A,x} \\ {}^{B}i_{A,y} & {}^{B}j_{A,y} & {}^{B}k_{A,y} \\ {}^{B}i_{A,z} & {}^{B}j_{A,z} & {}^{B}k_{A,z} \end{pmatrix} \begin{pmatrix} {}^{A}i_{o} & {}^{A}j_{o} & {}^{A}k_{o} \end{pmatrix}
$$

$$
{}^{B}R_{o} = {}^{B}R_{A}\;{}^{A}R_{o}
$$

The other way around: Given O in B, Find O in A



Coordinate Transformations & Rigid Body Motion

Generalizing – a point, O, with coordinates in Frame B can be transformed to coordinates in Frame A if the relative orientation between A & B (${}^{A}R_{B}$) is known

Coordinates of O in B

$${}^{A}P_{o} = {}^{A}R_{B}\, {}^{B}P_{o}$$

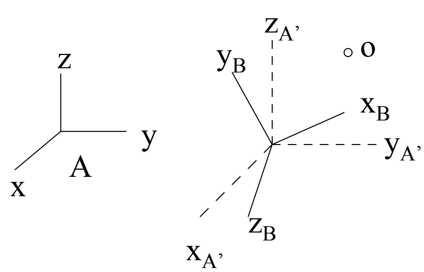
Is this always true?

Coordinates of O in A

Note Assumption : – origins of Frames A & B are coincident
– only difference is in orientation

Coordinate Transformations & Rigid Body Motion

What if the two origins are not coincident?
– i.e. , translation between frames A & B



A' = frame parallel to A
(only different origins)

$${}^{A'}P_{o} = {}^{A'}R_{B}\, {}^{B}P_{o} \qquad \text{(from before)}$$

Coordinate Transformations & Rigid Body Motion

${}^{A}P_o = {}^{A}P_B + {}^{A'}P_o$ } can only add two position vectors if they are expressed in Frames that are parallel to each other

A is parallel to A' – so that adding respective coordinates make sense

Therefore

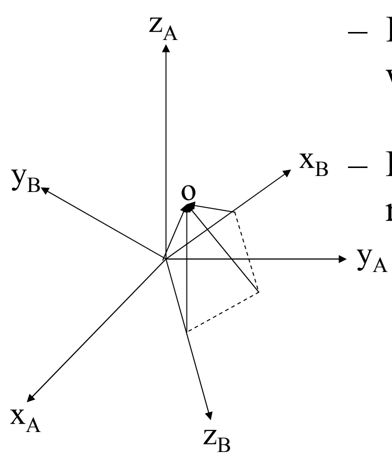
$$ {}^{A}P_o = \underbrace{{}^{A}P_B + \underbrace{{}^{A}R_B\,{}^{B}P_o}_{\text{Transforming O to a frame // to A}}}_{\text{Then adding the relative displacement of A \& B}} $$

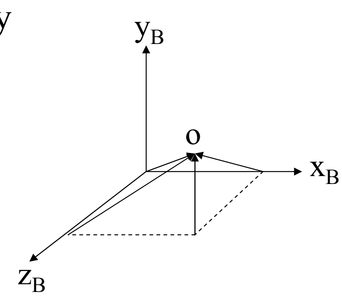
since ${}^{A}R_B = {}^{A'}R_B$ because A is // A'

Duality (of Coordinate Transformation) with Rigid Body Motion

– Imagine a Rigid Body (B=Pyramid) whose apex is at pt. O

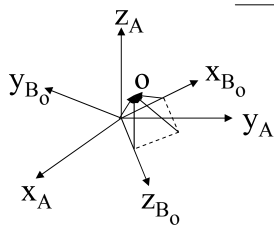
– Frame B is attached to the rigid body

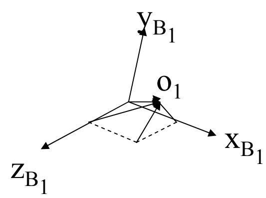




Duality (of Coordinate Transformation) with Rigid Body Motion

- The body B is initially at ${}^{A}R_{Bo}$ and ${}^{A}P_{o} = {}^{A}R_{Bo}\,{}^{B}P_{o}$
- Body B undergoes a motion such that the <u>new Position & Orientation</u> of B are $\underline{{}^{A}R_{B_1}}$ & $\underline{{}^{A}P_{B_1}}$

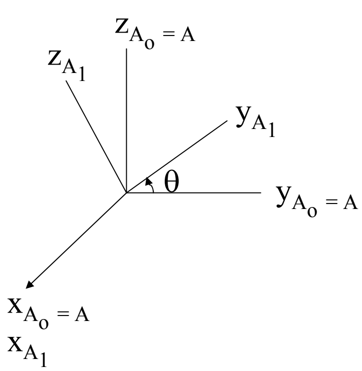




- New Coordinates of O in A, ${}^{A}P_{O_1}$ can be found :

$$ {}^{A}P_{o_1} = {}^{A}P_{B_1} + {}^{A}R_{B_1}\,{}^{B_1}P_{o1} $$

Elementary (Basic, Fundamental) Motions



- Initially, Body is at A_o
- A undergoes a Rotation about x_{Ao} by θ & A is now at A_1 after the rotation

$$ {}^{Ao}R_{A1} = \mathrm{Rot}\,(x,\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} $$

Similarly, a Rotation about y by θ :

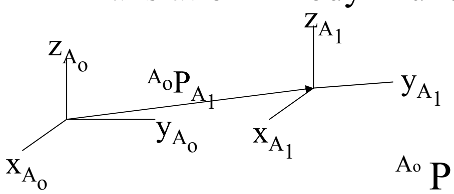
$$ {}^{Ao}R_{A1} = \mathrm{Rot}\,(y,\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} $$

Elementary (Basic, Fundamental) Motions

Similarly, a rotation about z by θ :

$$^{Ao}R_{A1} = \mathrm{Rot}\,(z,\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Translation – Body A undergoes pure translation (only)



$$^{Ao}P_{A1} = \begin{pmatrix} ^{Ao}P_{A1,x} \\ ^{Ao}P_{A1,y} \\ ^{Ao}P_{A1,z} \end{pmatrix}$$

Coordinates of A_1 in A_o

Homogeneous Transformations

- created so that rotations and translations are treated uniformly. (i.e. as matrix multiplications)
- vectors have 4 components
  - ~ scaling factor = 0 (free vectors, unit vectors along axis, orientation vectors)
  - ~ scaling factor = 1 (position vectors)

$$^{A}P_{B} = \begin{bmatrix} ^{A}P_{Bx} \\ ^{A}P_{By} \\ ^{A}P_{Bz} \\ 1 \end{bmatrix} \qquad ^{A}i_{B} = \begin{bmatrix} ^{A}i_{Bx} \\ ^{A}i_{By} \\ ^{A}i_{Bz} \\ \phi \end{bmatrix}$$

; similarly with j & k

Homogeneous Transformations

$$\begin{bmatrix} {}^{A}P_{C} \\ \hdashline 1 \end{bmatrix} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}P_{B} \\ \hdashline 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{B}P_{C} \\ \hdashline 1 \end{bmatrix}$$

$$\Rightarrow \quad {}^{A}P_{C} = {}^{A}R_{B}{}^{B}P_{C} + {}^{A}P_{B}$$

$${}^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}P_{B} \\ \hdashline 0 & 1 \end{bmatrix} \in \Re^{4\times 4}$$

= homogeneous transformation matrix

= describes the position and orientation of frame B in frame A

<u>Chain rule</u> :

$${}^{A}T_{C} = {}^{A}T_{B}{}^{B}T_{C}$$

Inverse of a Homogeneous Transformation Matrix

Given ${}^{A}T_{B}$, Find ${}^{B}T_{A}$

$${}^{A}T_{B}\,{}^{B}T_{A} = I$$

$$\begin{bmatrix} {}^{A}R_{B} & {}^{A}P_{B} \\ \hdashline \underline{0} & 1 \end{bmatrix} \begin{bmatrix} X & Y \\ \hdashline \underline{0} & 1 \end{bmatrix} = \begin{bmatrix} I & \underline{0} \\ \hdashline \underline{0} & I \end{bmatrix}$$

${}^{A}R_{B}\, X = I \quad \Rightarrow \quad X = {}^{A}R_{B}^{-1} = {}^{A}R_{B}^{T} = {}^{B}R_{A}$

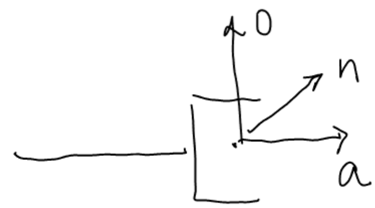
${}^{A}R_{B}\, Y + {}^{A}P_{B} = \underline{0}$

${}^{A}R_{B}\, Y = -{}^{A}P_{B}$

$Y = -{}^{A}R_{B}^{-1}\,{}^{A}P_{B} = -{}^{A}R_{B}^{T}\,{}^{A}P_{B}$

Inverse of a Homogeneous Transformation Matrix

$$\therefore {}^{A}T_{B}^{-1} = \left[\begin{array}{c:c} {}^{A}R_{B}^{T} & -{}^{A}R_{B}^{T}\ {}^{A}P_{B} \\ \hdashline \underline{0} & 1 \end{array}\right]$$



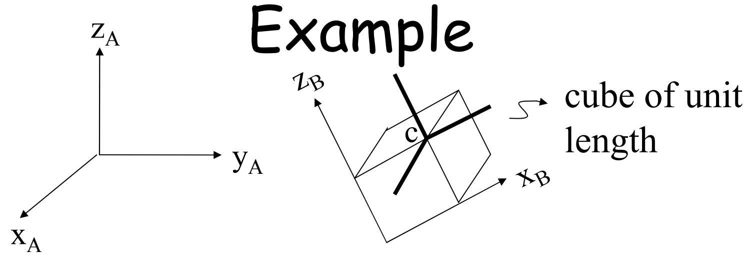
Note that

$$(\underline{n}\ \ \underline{0}\ \ \underline{a})^{T} = {}^{A}R_{B}^{T}$$

$${}^{A}R_{B}^{T}\ {}^{A}P_{B} = \begin{array}{l} \underline{n} \bullet {}^{A}P_{B} \\ \underline{0} \bullet {}^{A}P_{B} \\ \underline{a} \bullet {}^{A}P_{B} \end{array}$$

x, n = normal
y, o = opening
z, a = approach

Example



Given : Initial Position & Orientation of cube specified by ${}^{A}T_{B}$

Find : new coordinates of pt. c (corner of cube) after the cube is rotated by θ about z_A

Example :

$$ {}^{A}P_{C} = \underbrace{\mathrm{Rot}(z, 90^\circ)\ {}^{A}T_{B}}_{{}^{A}P_{B'}} \underbrace{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}}_{{}^{B'}P_{C} = {}^{B''}P_{C}} $$

since C is attached rigidly to B

Two Rules

- Given Bodies A and B, with ${}^{A}T_{B}$
- Body B moves, new position and orientation ${}^{A}T_{B1}$
  - with respect to A : Pre-multiplication
    - ${}^{A}T_{B1}$ = Motion (4x4) ${}^{A}T_{B}$
  - With respect to B (body itself): Post multiplication
    - ${}^{A}T_{B1}$ = ${}^{A}T_{B}$ Motion (4x4)

Other Orientation Representations

- Relative orientation between two bodies A and B can **always** be described by a sequence of 3 rotations about adjacent axes that are orthogonal to each other
- The 3 angles of rotation are used to represent orientation, together with their meanings (axes of rotation)

$$^{A}R_{B} = Rot(x,d)Rot(y,e)Rot(x,f)$$

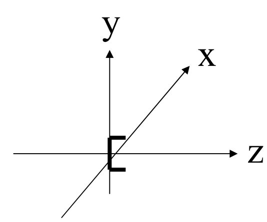
$$^{A}R_{B} = Rot(x,a)Rot(y,b)Rot(z,c)$$

$$^{A}R_{B} = Rot(z,p)Rot(y,q)Rot(z,r)$$

Many possibilities

- Given a definition or sequence of axes of rotations, are the three angles unique?

ROLL - PITCH - YAW Angles (ϕ, θ, φ)



$$ {}^{A}R_{B} = \underset{\text{Roll}}{\underset{\uparrow}{\mathrm{Rot}\,(z, \phi)}}\ \underset{\text{Pitch}}{\underset{\uparrow}{\mathrm{Rot}\,(y, \theta)}}\ \underset{\text{Yaw}}{\underset{\uparrow}{\mathrm{Rot}\,(x, \varphi)}} $$

$$ = \begin{pmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta\sin\varphi - \sin\phi\cos\varphi & \cos\phi\sin\theta\cos\varphi + \sin\phi\sin\varphi \\ \sin\phi\cos\theta & \sin\phi\sin\theta\sin\varphi + \cos\phi\cos\varphi & \sin\phi\sin\theta\cos\varphi - \cos\phi\sin\varphi \\ -\sin\theta & \cos\theta\sin\varphi & \cos\theta\cos\varphi \end{pmatrix} $$

Inverse Transformation

If $|n_z| \neq 1$

$$\phi = \mathrm{ATAN2}(n_y, n_x) \quad \theta = \mathrm{ATAN2}(-n_z, \pm\sqrt{n_x^2 + n_y^2}) \quad \varphi = \mathrm{ATAN2}(o_z, a_z)$$

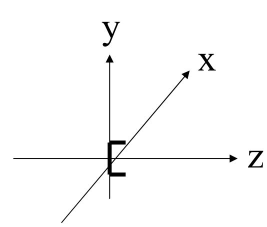
(2 Solutions)

If $n_z = +1$, $\theta = 270°$, $\varphi + \phi = \mathrm{ATAN2}(-o_x, o_y)$

If $n_z = -1$, $\theta = 90°$, $\varphi - \phi = \mathrm{ATAN2}(o_x, o_y)$

Mathematical Singularity occurs when $|n_z| = 1$
φ & ϕ describe the same rotation & cannot be computed separately

ROLL - PITCH - YAW Angles (ϕ, θ, φ)



$$^{A}R_{B} = \underbrace{\mathrm{Rot}(z, \phi)}_{\text{Roll}} \; \underbrace{\mathrm{Rot}(y, \theta)}_{\text{Pitch}} \; \underbrace{\mathrm{Rot}(x, \varphi)}_{\text{Yaw}}$$

$$= \begin{pmatrix} 0 & -\sin(\phi-\varphi) & \cos(\phi-\varphi) \\ 0 & \cos(\phi-\varphi) & -\sin(\phi-\varphi) \\ -1 & 0 & 0 \end{pmatrix}$$

At pitch = 90 deg

Roll and Yaw describes the same motion