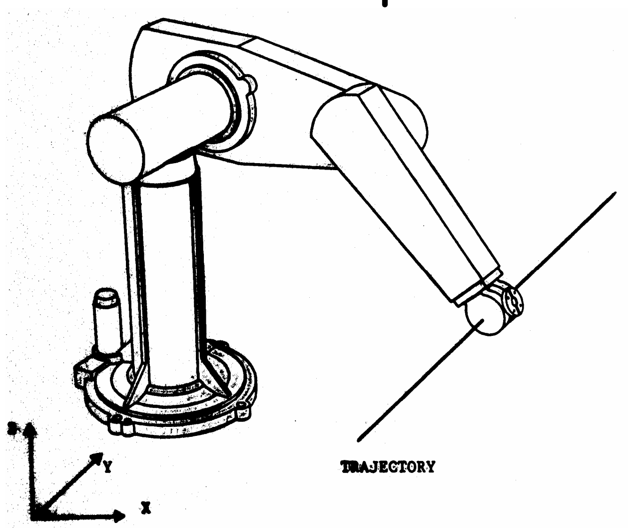
CHAPTER 2

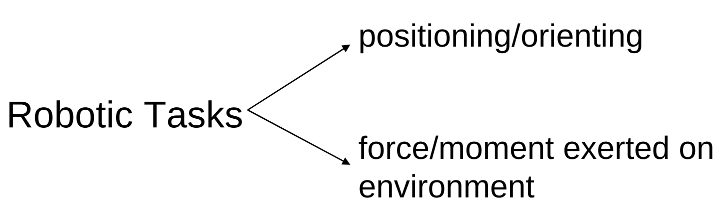
Robot Kinematics of Position

Learning Objectives

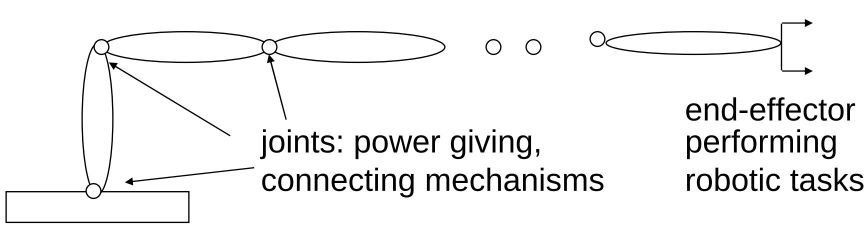
- Given a robot, derive a kinematic model of the robot
  - Assign frames
  - Derive equations relating relative position and orientation of frames (forward and inverse equations)

Robotic Manipulator



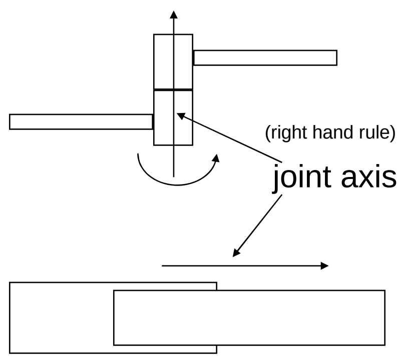


Chain of rigid bodies connected by joints



Robot Joints

Two Basic Types:



Rotational

Translational
(Prismatic)

Degrees-of-Freedom

3D Space = 6 DOF → 3 position
→ 3 orientation

In robotics,

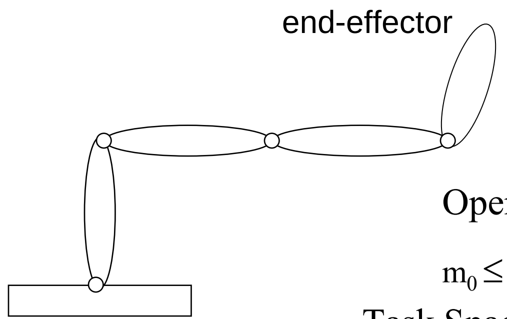
DOF = number of independently driven joints

As DOF ↑
- positioning accuracy ↓
- computational complexity ↑
- cost ↑
- flexibility ↑
- power transmission is more difficult

Cartesian Space:

m parameters
6 independent parameters

To completely
specify: $6 \leq m$



Operational Space:

$m_0 \leq 6$ independent parameters

Task Space:

$m_k < m$: subset of end-effector parameters to accomplish the task
$m_{k(0)}$: if independent parameters

Joint Space:

n DOF

Robot Kinematic Modeling

<u>THE DENAVIT-HARTENBERG REPRESENTATION</u>

In the robotics literature, the Denavit-Hartenberg (D-H) Representation has been used, almost universally, to derive the kinematic description of robotic manipulators. The appeal of the D-H representation lies in its algorithmic approach. In this handout, we provide an algorithm for the assignment of robotic coordinate frames, highlight the conventions associated with the D-H approach, and exemplify the development through the Puma and Stanford manipulators.

Robot Kinematic Modeling

STEP 1 : Number the Robot Joints and Links

Robotic manipulators are articulated, open kinematic chains of N rigid bodies (links) which are connected serially by joints. The links are numbered consecutively from the base (link 0) to the final end (link N). The joints are the points of articulation between the links and are numbered from 1 to N so that joint i connects links (i-1) and i. Each joint provides one degree-of-freedom which can either be a rotation or translation.There is no joint at the end of the final link.

Robot Kinematic Modeling

STEP 2 : Assign Link Coordinate Frames

To describe the geometry of robot motion, we assign a Cartesian coordinate frame (O_i;x_i,y_i,z_i) to each link, as follows:

- the z_i axis is directed along the axis of motion of joint (i + 1), that is, link (i + 1) rotates about or translates along z_i;
- the x_i axis lies along the common normal from the z_{i-1} axis to the z_i axis (if z_{i-1} is parallel to z_i, then x_i is specified arbitrarily, subject only to x_i being perpendicular to z_i); and

Robot Kinematic Modeling

STEP 2 : Assign Link Coordinate Frames

- the y_i axis completes the right-handed coordinate system.

The origin of the robot base frame O_0 can be placed anywhere in the supporting base and the origin of the last (end-effector) coordinate frame O_N is specified by the geometry of the end-effector.

Robot Kinematic Modeling

STEP 3 : Define the Joint Coordinates

The joint coordinate q_i is the angular displacement around z_{i-1} if joint *i* is rotational, or the linear displacement along z_{i-1} if joint *i* is translational. The N-dimensional space defined by the joint coordinates $(q_1,\ldots,q_N)$ is called the configuration space of the N DOF mechanism.

Robot Kinematic Modeling

STEP 4 : Identify the Link Kinematic Parameter

In general, four elementary transformations are required to relate the *i*-th coordinate frame to the (*i*-1)-th coordinate frame:

- Rotate an angle of θ_i (in the right-handed sense) about the z_{i-1} axis, so that the x_{i-1} axis is parallel to the x_i axis.
- Translate a distance of r_i along the positive direction of the z_{i-1} axis, to align the x_{i-1} axis with the x_i axis.

Robot Kinematic Modeling

STEP 4 : Identify the Link Kinematic Parameter

- Translate a distance of d_i along the positive direction of the $x_{i-1} = x_i$ axis, to coalesce the origins O_{i-1} and O_i.
- Rotate an angle of α_i (in the right-handed sense) about the $x_{i-1} = x_i$ axis, to coalesce the two coordinate systems.

The *i*-th coordinate frame is therefore characterized by the four D-H kinematic link parameters θ_i, r_i, d_i and α_i. If joint *i* is rotational, then $q_i = \theta_i$, and α_i, d_i and r_i are constant parameters which depend upon the

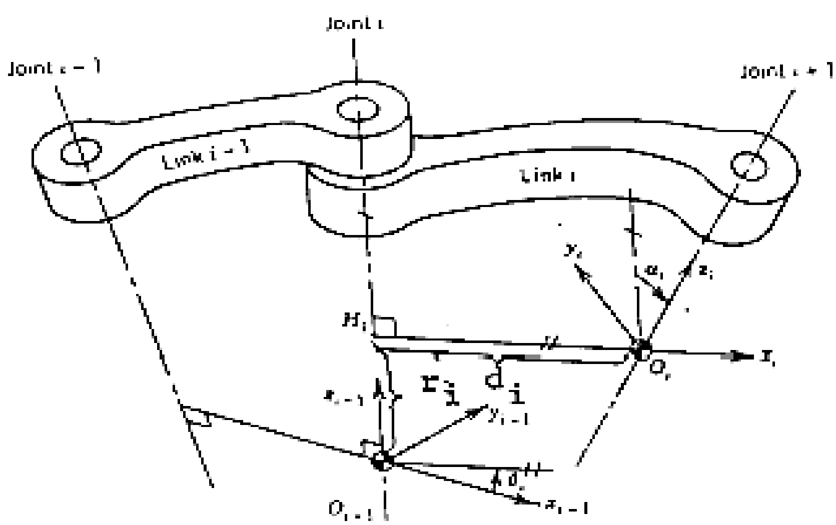
Robot Kinematic Modeling

STEP 4 : Identify the Link Kinematic Parameter

geometric properties and configuration of link *i*. If joint *i* is translational, then $q_i = r_i$, and d_i, α_i and θ_i are constant parameters which depend upon the configuration of link *i*. For both rotational and translational joints, r_i and θ_i are the distance and angle between links $(i - 1)$ and *i*; d_i and α_i are the length and twist of link *i*.

Robot Kinematic Modeling

STEP 4 : Identify the Link Kinematic Parameter



Robot Kinematic Modeling

STEP 5 : Define the Link Transformation Matrices

The position and orientation of the *i*-th coordinate frame can be expressed in the (i – 1)-th coordinate frame by the following homogeneous transformation matrix:

$$A_i = \mathrm{Rot}(z, \theta)\,\mathrm{Trans}(0, 0, r_i)\,\mathrm{Trans}(d_i, 0, 0)\,\mathrm{Rot}(x, \alpha)$$

$$A_i(q_i) = {}^{i\text{-}1}T_i = \begin{pmatrix} \cos\theta_i & -\cos\alpha_i \sin\theta_i & \sin\alpha_i \sin\theta_i & d_i \cos\theta_i \\ \sin\theta_i & \cos\alpha_i \cos\theta_i & -\sin\alpha_i \cos\theta_i & d_i \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & r_i \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Robot Kinematic Modeling

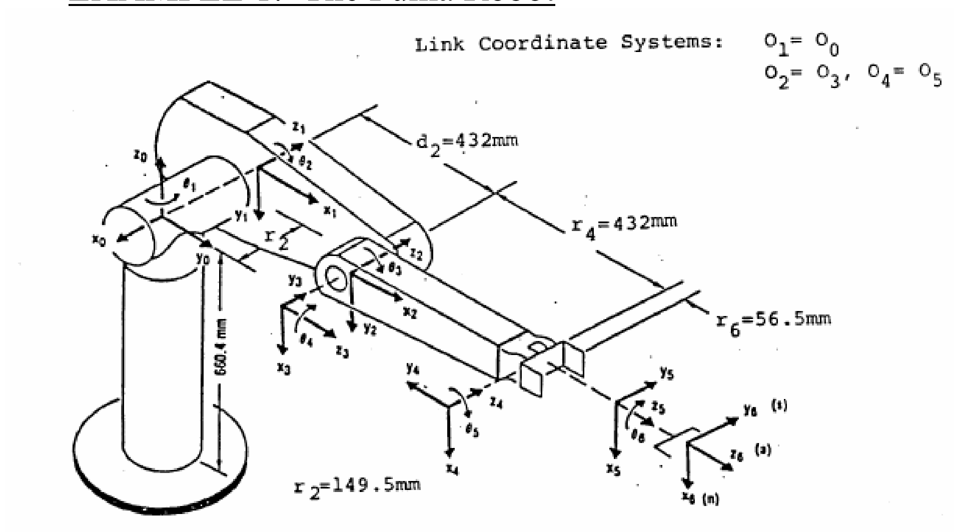
STEP 6 : Compute the Forward Transformation Matrix

The position and orientation of the end-effector coordinate frame is expressed in the base coordinate frame by the forward transformation matrix:

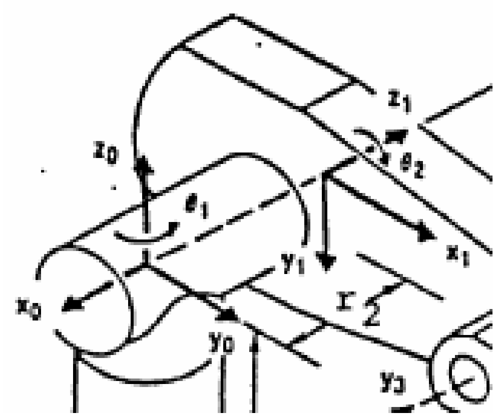
$${}^0T_N(q_1, q_2,\ldots, q_N) = {}^0T_N = A_1A_2\ldots A_N = \begin{pmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Robot Kinematic Modeling

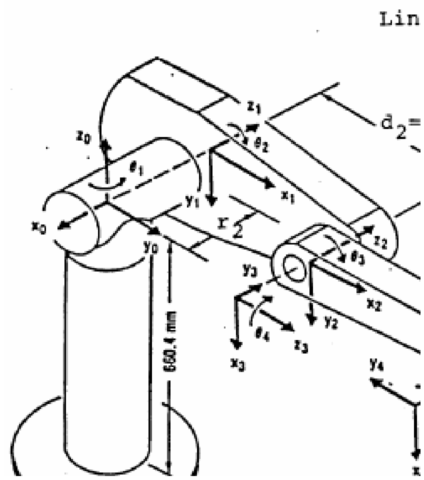
EXAMPLE 1: The Puma Robot



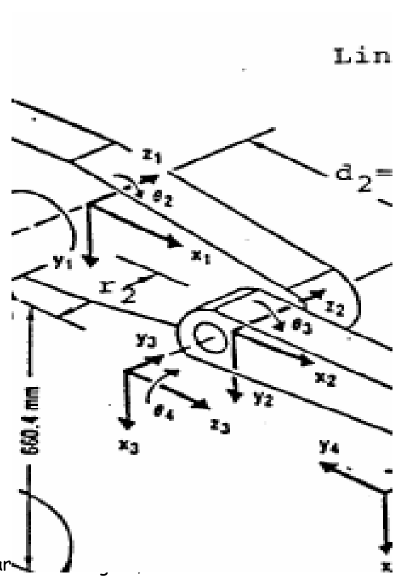
Frames 0 to 1



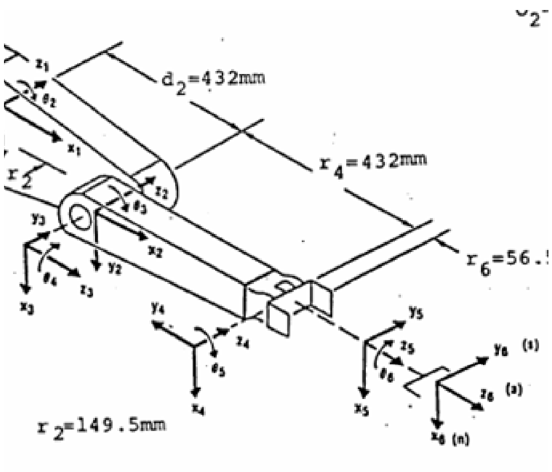
Frames 1 to 2



Frames 2 to 3

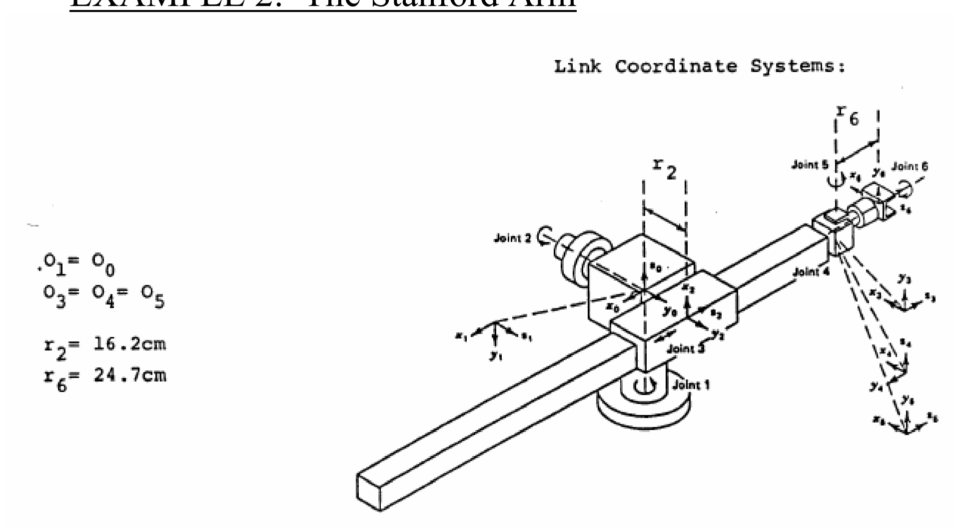


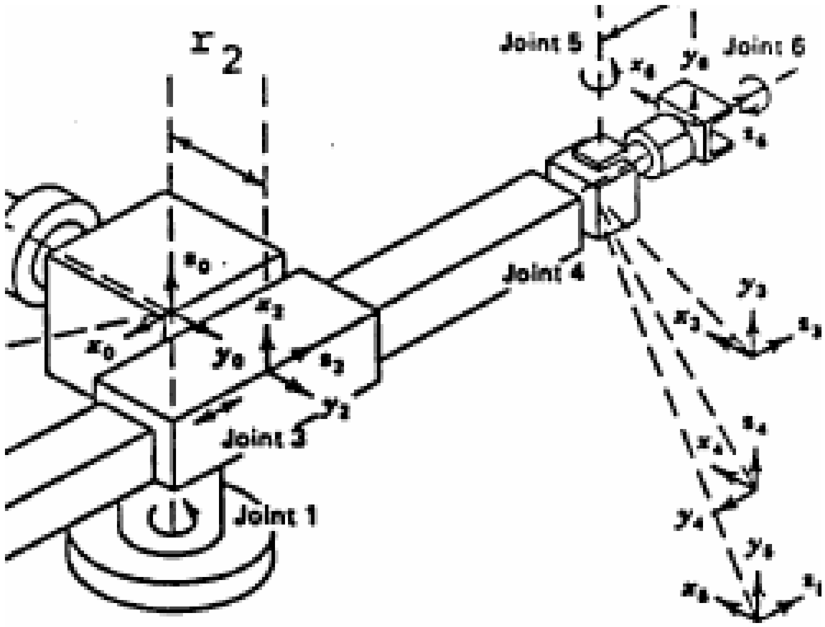
Frames 3 to 4, to 5, to 6



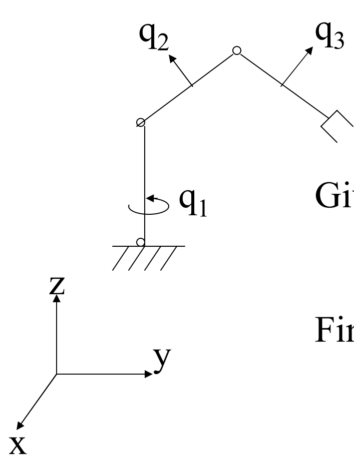
Robot Kinematic Modeling

EXAMPLE 2: The Stanford Arm





Forward Kinematic Problem



Given: q1, q2, q3….
(n joint positions)

Find: End-Effector position P_E
and orientation R_E
(m end-effector parameters)

Forward Kinematic Problem

1. Assign Cartesian Coordinate frames to each link (including the base ϕ & end-effector N)

2. Identify the joint variables and link kinematic parameters

3. Define the link transformation matrices. ${}^{i-1}T_i = A_i$

4. Compute the forward transformation
$${}^{0}T_N(q_1, q_2, \ldots, q_N) = A_1 A_2 A_3 \ldots A_N = \begin{pmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Inverse Kinematic Problem

Given: Position & Orientation of END-EFFECTOR

Find: joint coordinates

$$\underline{{}^{0}T_N} \longrightarrow q_1, q_2, q_3, \ldots, q_N$$

Need to solve at most six independent equations in N unknowns.

Inverse Kinematic Problem

<u>ISSUES</u>

- Existence of solutions
  - Workspace
  - Dextrous Workspace
  - Less than 6 joints
  - Joint limits (practical)
- Multiple solutions
  - Criteria
  - Solvability
    - closed form
      - Algebraic
      - Geometric
    - numerical
  - number of solutions
    = 16 $\quad d_i, r_i \neq 0$ $\quad$ <u>for six points</u>

Solution To Inverse Kinematics

$$^0T_N = {}^0T_1\,{}^1T_2\,{}^2T_3 \ldots {}^{N-1}T_N = A_1A_2A_3\ldots A_N$$

$$\text{Given: } {}^0T_N = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ \phi & \phi & \phi & 1 \end{bmatrix} \quad A_i = \begin{bmatrix} c\theta_i & -c\alpha_i s\theta_i & s\alpha_i s\theta_i & d_i c\theta_i \\ s\theta_i & c\alpha_i c\theta_i & -s\alpha_i c\theta_i & d_i s\theta_i \\ \phi & s\alpha_i & c\alpha_i & r_i \\ \phi & \phi & \phi & 1 \end{bmatrix}$$

Find: $q = q_1, q_2, q_3, \ldots, q_N$ (joint coordinates)

Solution To Inverse Kinematics

$$\begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ \phi & \phi & \phi & 1 \end{bmatrix} = A_1 A_2 A_3 \ldots A_N$$

12 Equations — 6 independent, 6 redundant

N unknowns

LHS(i,j) = RHS(i,j)

row: i = 1, 2, 3

column: j = 1, 2, 3, 4

Solution To Inverse Kinematics

General Approach: Isolate one joint variable at a time

$$\underbrace{A_1^{-1}\,{}^0T_N}_{\text{function of } q_1} = A_2 A_3 \ldots A_N = \underbrace{{}^1T_N}_{\text{function of } q_2, \ldots, q_N}$$

- Look for constant elements in ${}^1T_N$
- Equate LHS(i,j) = RHS(i,j)
- Solve for q_1

Solution To Inverse Kinematics

$$\underbrace{A_2^{-1}A_1^{-1}{}^0T_N}_{\text{function of } q_1,\, q_2} = A_3 \ldots A_N = \underbrace{{}^2T_N}_{\text{function of } q_3, \ldots, q_N}$$

only one unknown q_2 since q_1 has been solved for

- Look for constant elements of ${}^2T_N$
- Equate LHS(i,j) = RHS(i,j)
- Solve for q_2
- Maybe can find equation involving q_1 only

Note:
- There is no algorithmic approach that is 100% effective
- Geometric intuition is required

Solution To Inverse Kinematics

There are Two Classes of Robot Geometries for which closed-form inverse kinematic solutions are guaranteed. They are:

1. Robots with any 3 joints TRANSLATIONAL

2. Robots with any 3 rotational joint axes co-intersecting at a common point

These are DECOUPLED ROBOT GEOMETRIES

meaning

- can reduce system to a lower order subsystem (i.e. 3rd-order) for which closed form solutions are guaranteed

General Analytical Inverse Kinematic Formula

Case 1: $\sin\theta = a$, $\cos\theta = b$; $a \in [-1,1]$, $b \in [-1,1]$ → $\theta = \text{ATANZ}(a, b)$

unique

Case 2: $\sin\theta = a$, $a \in [-1,1]$, $\cos\theta = \pm\sqrt{1-a^2}$ → $\theta = \text{ATANZ}(a, \pm\sqrt{1-a^2})$

2 solutions

θ, $180^o - \theta$

@ $\theta = \pm 90^o$, $|a| = 1$,

"boundary" → singularity

$\cos\theta = b$, $b \in [-1,1]$, $\sin\theta = \pm\sqrt{1-b^2}$ → $\theta = \text{ATANZ}(\pm\sqrt{1-b^2}, b)$

2 solutions

θ, $-\theta$

@ $\theta = 0^o, 180^o$, $|b| = 1$,

degeneracy of order 2 "boundary" → singularity

General Analytical Inverse Kinematic Formula

Case 3: $a\cos\theta + b\sin\theta = 0 \rightarrow \theta = \text{ATANZ}(a, -b)$ or $\text{ATANZ}(-a, b)$

2 solutions, 180^o apart

Singularity when $a = b = 0$

→ infinite order degeneracy

Case 4: $a\cos\theta + b\sin\theta = c$ $a, b, c \neq 0$ 2 solutions

$$\theta = \text{ATANZ}(b, a) + \text{ATANZ}(\pm\underbrace{\sqrt{a^2+b^2-c^2}}_{\geq 0 \text{ For solution to exist}}, c)$$

$a^2 + b^2 + c^2 < 0$ → outside workspace

$a^2 + b^2 + c^2 = 0$ → 1 solution (singularity)

degeneracy of order 2

General Analytical Inverse Kinematic Formula

Case 5: $\sin\theta\sin\phi = a$
$\cos\theta\sin\phi = b$

$\theta = \text{ATANZ}(a, b)$ if $\sin\phi$ is $\oplus$ positive
$\theta = \text{ATANZ}(-a, -b)$ if $\sin\phi$ is $\ominus$ negative

If $\cos\phi = c \rightarrow \phi = \text{ATANZ}(\pm\sqrt{a^2+b^2}, c)$ (2 solutions for ϕ)

Then 2 solutions:

$\theta = \text{ATANZ}(a, b)$	$\theta = \text{ATANZ}(-a, -b)$
$\phi = \text{ATANZ}(\sqrt{a^2+b^2}, c)$	$\phi = \text{ATANZ}(-\sqrt{a^2+b^2}, c)$

Singularity: $a = b = 0$ $|c| = 1$
θ = undefined ϕ = 1 solution

General Analytical Inverse Kinematic Formula

Case 6: $a\cos\theta - b\sin\theta = c$ (1)
$a\sin\theta + b\cos\theta = d$ (2)

Then $\theta = \text{ATANZ}(ad - bc, ac + bd)$
1 solution

Note that for (1) & (2) to be satisfied, or at (1) & (2), we have

$$a^2 + b^2 = c^2 + d^2$$

Decoupling (Kinematic)

"Finding a <u>subset of joints</u> primarily responsible for the completion of a <u>subset of the manipulator task</u>"

Involves the identification of:

- decoupled task ← Total Task
- decoupled robot subsystem responsible for the decoupled task

Decoupled Robot Geometry – refers to a manipulator Geometry for which decoupling is guaranteed

Decoupling (Kinematic)

Decoupled Robot Geometries: (6-axes)

1. Any Three (3) Translational Joints
2. Any Three Co-Intersecting Rotational Axes

Identified by Pieper, 1968

3. Any 2 Transl. Joints Normal to a Rot. Joint
4. Transl. Joint Normal to 2 Parallel Joints
5. Any 3 Rot, Joints Parallel

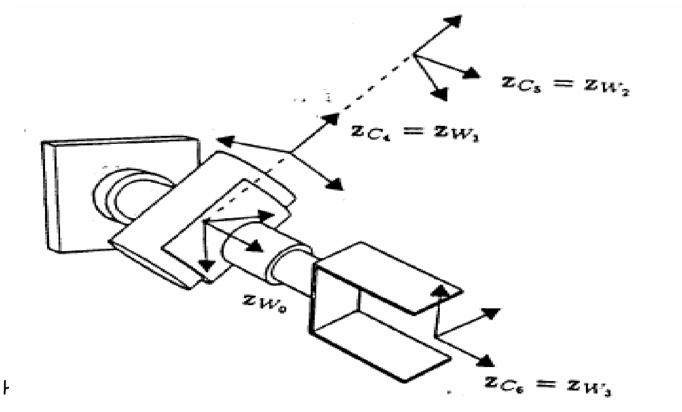
New geometries Identified by Ang, 1992*

V.D. Tourassis and M.H. Ang Jr., "Task Decoupling in Robot Manipulators," Journal of Intelligent and Robotic Systems 14:283-302, 1995. (Technical Report in 1992).

Decoupling (Kinematic)

Robots with Spherical Wrists is a popular decoupled robot geometry

→ 3 wrist axes co-intersecting at a common point



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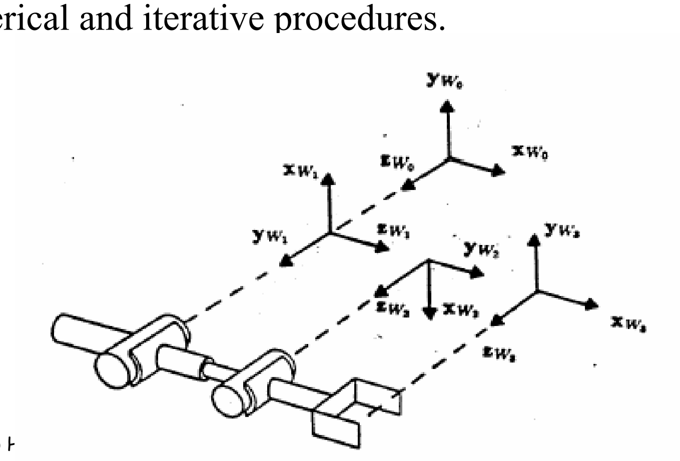
Decoupling (Kinematic)

For robots that do not have decoupled geometries, a closed Form solution may not exist, → one has to resort to numerical and iterative procedures.



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Numerical Solutions

- m equations in n unknowns
- start with an initial estimate for the n unknowns
- compute the error caused by this inaccurate estimate

$^{D}T_N = (T_D)^{-1} T_N$ = position & orientation of end-effector frame with respect to origin of target frame

$\Updownarrow$

$r_x \; r_y \; r_z \; r_\phi \; r_\theta \; r_\varphi$

- modify estimate to reduce error

Numerical Solutions

Three important requirements for the numerical algorithm are:

i. a priori conditions for convergence
ii. insensitivity to initial estimates
iii. provision for multiple solutions

- The most common methods are based on the Newton-Raphson approach.

Ref: A.A.Goldenberg, B. Benhabib, & R.G.Fenton, "A Complete Generalized Solution to the Inverse Kinematics of Robots" IEEE Journal of Rob. & Auto. 1(1): March 1985, pp. 14-20.