CHAPTER 3

Rigid Body Motion and
Robot Kinematics of Velocity

Learning Objectives

1. Relate time derivatives of position and orientation representations with translational and angular velocities.
2. Transform velocities in different spaces
3. Relate joint velocities with end-effector velocities
4. Understand robot singularities
5. Use velocity relationships to command motion of robots (resolved rate motion control)

Translational Velocities

${}^{A}\mathbf{u}_{B} \in \Re^{3x1}$ = translational velocity of frame B (i.e., origin of frame B) relative of frame A

$$ {}^{A}\mathbf{u}_{B} = \frac{d}{dt}\,{}^{A}\mathbf{p}_{B} = \lim_{\Delta t \to 0} \frac{{}^{A}\mathbf{p}_{B}(t+\Delta t) - {}^{A}\mathbf{p}_{B}(t)}{\Delta t} $$

"frame of differentiation" is A

Velocity, like any vector may be expressed in another frame, say W

$$ {}^{W}_{A}u_{B} = {}^{W}R_{A}\,{}^{A}u_{B} $$

Translational Velocities

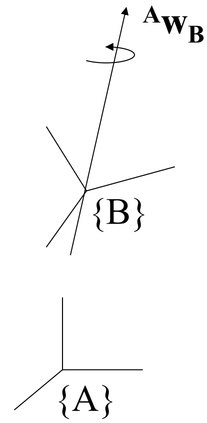
In general, velocity vector depends on 2 frames:

- frames of differentiation – A – leading subscript – this is the frame where the velocity of B is instantaneously computed from (can be thought of as velocity reference point)
- frame resulting vector is expressed in – W – leading superscript

When leading subscript is omitted, it is implied that the velocity is relative to some understood universal frame of reference.
A missing leading superscript implies a generic frame of expression.

Rotational Velocities

- At a certain instant, frame B has an orientation $^{A}\mathbf{R_B}$ and its rotational motion may be represented by the rotational (angular) velocity vector



$^{A}\mathbf{w_0} \in \Re^{3x1}$

unit vector along $^{A}\mathbf{w_B}$ = instantaneous axis of rotation = $^{A}\mathbf{k_B}$

magnitude of $^{A}\mathbf{w_B}$ = speed of rotation

- $^{A}\mathbf{w_B}$ is related to $\frac{d}{dt}{}^{A}\mathbf{R}_B = {}^{A}\dot{\mathbf{R}}_B = \lim_{\Delta t \to 0} \frac{^{A}\mathbf{R_B}(t+\Delta t) - {}^{A}\mathbf{R_B}(t)}{\Delta t}$

Rotational Velocities

Let **Rot (k, dθ)** = incremental change in rotation.

$$^{A}\mathbf{w_B} = {}^{A}\mathbf{k_B}\,\dot{\theta} = {}^{A}\mathbf{k_B}\,\frac{d\theta}{dt}$$

$$\therefore\ {}^{A}\mathbf{R_B} + \Delta\, {}^{A}\mathbf{R_B} \underbrace{= \mathbf{Rot\,(\,k\,,\,d\theta\,)}}_{} {}^{A}\mathbf{R_B}$$

pre-multiplication since $d\theta\, {}^{A}\vec{\mathbf{k}}_B$ is expressed in the frame A (base)

Rot (k, θ) =

$$\begin{pmatrix} k_x k_x \operatorname{vers}\theta + \cos\theta & k_y k_x \operatorname{vers}\theta - k_z \sin\theta & k_z k_x \operatorname{vers}\theta + k_y \sin\theta \\ k_x k_y \operatorname{vers}\theta + k_z \sin\theta & k_y k_y \operatorname{vers}\theta + \cos\theta & k_z k_y \operatorname{vers}\theta - k_x \sin\theta \\ k_x k_z \operatorname{vers}\theta - k_y \sin\theta & k_y k_z \operatorname{vers}\theta + k_x \sin\theta & k_z k_z \operatorname{vers}\theta + \cos\theta \end{pmatrix}$$

where versθ = (1 - cosθ)

Rotational Velocities

For a differential change (small) <u>dθ</u>

$$\cos(\theta + d\theta) = \cos\theta + \frac{\delta\cos\theta}{\delta\theta}d\theta = \cos\theta + (-\sin\theta)d\theta$$

for $\theta = 0$, $d\theta$ small

$$\cos(d\theta) \overset{\rightarrow}{=} 1 \quad \text{(approx)}$$

$$\sin(\theta + d\theta) = \sin\theta + \frac{\delta\sin\theta}{\delta\theta}d\theta = \sin\theta + \cos\theta(d\theta)$$

Rotational Velocities

for $\theta = 0$, $d\theta$ = small

$$\sin(d\theta) = 0 + 1d\theta$$

$$\overset{\rightarrow}{=} d\theta \quad \text{(approx)}$$

for $\theta = 0$, $d\theta$ = small

$$\text{vers}(d\theta) = 1 - \cos(d\theta) \overset{\rightarrow}{=} 0 \quad \text{(approx)}$$

Rotational Velocities

$$\therefore \mathbf{Rot}\ (\mathbf{k}, \mathbf{d\theta}) = \begin{bmatrix} 1 & -k_z d\theta & k_y d\theta \\ k_z d\theta & 1 & -k_x d\theta \\ -k_y d\theta & k_x d\theta & 1 \end{bmatrix}$$

Back to:

$$\Delta {}^A R_B = \mathrm{Rot}\,(k, d\theta)\ {}^A R_B - {}^A R_B$$

$$= \left[\mathrm{Rot}\,(k, d\theta) - I\right]\ {}^A R_B$$

$$\Delta {}^A R_B = \begin{bmatrix} \phi & -k_z d\theta & k_y d\theta \\ k_z d\theta & \phi & -k_x d\theta \\ -k_y d\theta & k_x d\theta & \phi \end{bmatrix} {}^A R_B$$

Rotational Velocities

dividing by dt:

$$\frac{d^A\mathbf{R_B}}{dt} = \begin{bmatrix} \phi & -k_z \frac{d\theta}{dt} & k_y \frac{d\theta}{dt} \\ k_z \frac{d\theta}{dt} & \phi & -k_x \frac{d\theta}{dt} \\ -k_y \frac{d\theta}{dt} & k_x \frac{d\theta}{dt} & \phi \end{bmatrix} {}^A\mathbf{R_B}$$

But $${}^A\mathbf{w}_B = {}^A\mathbf{k}_B\ \dot{\theta} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \dot{\theta} = \begin{bmatrix} {}^A w_{Bx} \\ {}^A w_{By} \\ {}^A w_{Bz} \end{bmatrix}$$

Rotational Velocities

$$\therefore\ {}^{A}\dot{\mathbf{R}}_{\mathbf{B}} = \underbrace{\begin{bmatrix} 0 & -{}^{A}w_{Bz} & {}^{A}w_{By} \\ {}^{A}w_{Bz} & 0 & -{}^{A}w_{Bx} \\ -{}^{A}w_{By} & {}^{A}w_{Bx} & 0 \end{bmatrix}}\ {}^{A}\mathbf{R}_{\mathbf{B}}$$

Let this be ${}^{A}\widehat{\mathbf{w}}_{\mathbf{B}}$=
"cross product operator" of ${}^{A}\mathbf{w}_{\mathbf{B}}$

$${}^{A}\dot{\mathbf{R}}_{\mathbf{B}} = {}^{A}\widehat{\mathbf{w}}_{\mathbf{B}}\ {}^{A}\mathbf{R}_{\mathbf{B}}$$

The Orthonormal (Rotation) Matrix & Skew Symmetric Matrices

Skew symmetric matrix as vector cross product:

$$\text{Let } \mathbf{S} = \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix} \text{ and } \mathbf{w} = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$$

Then

$\mathbf{Sp} = \mathbf{w} \times \mathbf{p}$
where $\mathbf{p} \in \Re^{3x1}$ vector

3 x 3 matrix version of 3 x 1 vector
Cross product equivalent

$$\dot{R} = \hat{\omega} R \rightarrow \underbrace{\begin{pmatrix} \dot{x} & \dot{y} & \dot{z} \end{pmatrix}}_{\text{Time derivative of Unit vectors}} = \hat{\omega} \underbrace{\begin{pmatrix} x & y & z \end{pmatrix}}_{\text{3 x 1 unit vectors}}$$

$$\dot{x} = w \times x = \hat{w} x$$

$$\dot{y} = w \times y = \hat{w} y$$

$$\dot{z} = w \times z = \hat{w} z$$

Orientation Error

- Angular velocity – measure of instantaneous 3 x 1 orientation error
- Allows the relationship between

$$\Delta R \Leftrightarrow \Delta \Phi \qquad \Delta \Phi = \int \omega \qquad \Delta R = \int \dot{R}$$

$$\dot{R} = \hat{\omega} R \rightarrow \begin{pmatrix} \dot{x} & \dot{y} & \dot{z} \end{pmatrix} = \hat{\omega} \begin{pmatrix} x & y & z \end{pmatrix}$$

Rotational Velocities

As with any vector, the rotational velocity vector ${}^{A}\mathbf{w}_{B}$ may be expressed in another frame C:

$$\underbrace{{}^{C}_{A}\mathbf{w}_{B}} = {}^{C}\mathbf{R}_{A} \; {}^{A}\mathbf{w}_{B}$$

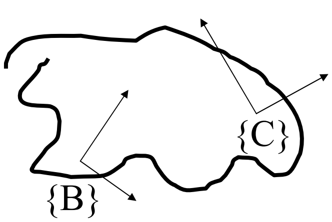
leading subscript A: frame the body is rotating relative to (frame & differentiation)

leading superscript C: frame of Expression

Rotational Velocities

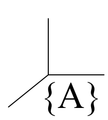
The equivalent matrix product representation is:

$${}^{C}_{A}\dot{W}_{B} = {}^{C}\mathbf{R}_{A} \, {}^{A}\mathbf{w}_{B} \; {}^{A}\mathbf{R}_{C}$$



{B} & {C} are attached to the same rigid body which is rotating

$${}^{A}\mathbf{R}_{C} = {}^{A}\mathbf{R}_{B} \, {}^{B}\mathbf{R}_{C}$$



Diff with time $\longrightarrow$ $${}^{A}\dot{\mathbf{R}}_{C} = {}^{A}\mathbf{R}_{B} \, \cancelto{\phi}{{}^{B}\dot{\mathbf{R}}_{C}} + {}^{A}\dot{\mathbf{R}}_{B} \, {}^{B}\mathbf{R}_{C}$$

Rotational Velocities

$$\underbrace{{}^{A}\hat{\mathbf{w}}_{C}\,{}^{A}\mathbf{R}_{C}}_{{}^{A}\dot{R}_{C}} = \underbrace{{}^{A}\hat{\mathbf{w}}_{B}\,{}^{A}\mathbf{R}_{B}}_{{}^{A}\dot{R}_{B}}\,{}^{B}\mathbf{R}_{C}$$

$${}^{A}\hat{\mathbf{w}}_{C}\,\cancel{{}^{A}\mathbf{R}_{C}} = {}^{A}\hat{\mathbf{w}}_{B}\,\cancel{{}^{A}\mathbf{R}_{C}}$$

$${}^{A}\mathbf{w}_{C} = {}^{A}\mathbf{w}_{B}$$

$\Rightarrow$ rotational velocity of rigid body is equal to rot. velocity of any frame attached to the rigid body.

The Orthonormal (Rotation) Matrix & Skew Symmetric Matrices

$\mathbf{R}\mathbf{R}^{\mathbf{T}} = \mathbf{I}$

$\mathbf{R}\dot{\mathbf{R}}^{\mathbf{T}} + \dot{\mathbf{R}}\mathbf{R}^{\mathbf{T}} = \underline{0}$

$(\dot{\mathbf{R}}\mathbf{R}^{\mathbf{T}})^{\mathbf{T}} + \dot{\mathbf{R}}\mathbf{R}^{\mathbf{T}} = \underline{0}$

Define $\mathbf{S} = \dot{\mathbf{R}}\mathbf{R}^{\mathbf{T}} = \hat{\mathbf{w}}$

Then $\mathbf{S} + \mathbf{S}^{\mathbf{T}} = \underline{0}$
$\mathbf{S}$ = a skew symmetric matrix

Roll Pitch Yaw Rates & Angular Velocities

$^{A}\mathbf{R_B} = \mathbf{Rot\ (\ z,\ \phi\ )\ Rot\ (\ y,\ \theta\ )\ Rot\ (\ x,\ \varphi)}$

$$w = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \dot{\phi} + \mathrm{Rot}(z,\phi) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \dot{\theta} + \mathrm{Rot}(z,\phi)\,\mathrm{Rot}(y,\theta) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \dot{\varphi}$$

$$w = \underbrace{\begin{pmatrix} 0 & -\sin\phi & \cos\phi\cos\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 1 & 0 & -\sin\theta \end{pmatrix}}_{\mathbf{E_r^{-1}}_{\mathbf{RPY}}} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\varphi} \end{pmatrix}$$

$\mathbf{w = E_r^{-1}\,\dot{x}_r}$

OR

$\mathbf{\dot{x}_r = E_r\,w}$

Roll Pitch Yaw Ratio & Angular Velocities

$$\dot{x}_r = \underbrace{\begin{pmatrix} \dfrac{\cos\phi\sin\phi}{\cos\theta} & \dfrac{\sin^2\phi}{\cos\theta} & 1 \\ -\sin\phi & \cos\phi & 0 \\ \dfrac{\cos\phi}{\cos\theta} & \dfrac{\sin\phi}{\cos\theta} & 0 \end{pmatrix}}_{} w$$

If $\cos\theta = 0$, matrix does not exist

→ Math. Singularity

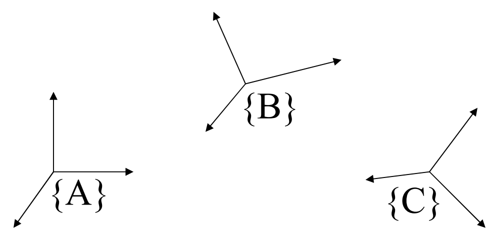
$\mathbf{w} \rightarrow \mathbf{\dot{x}_r}$ not always possible

Not all possible angular matrix can be represented

This is a problem with 3 parameter representations for orientation

Simultaneous Rotational & Translational Velocities

Given: Frames A, B & C



{B} & {C} in motion with respect to {A}

Find: Relationships between velocities

$$\mathbf{{}^{A}p_{C} = {}^{A}p_{B} + {}^{A}R_{B}\,{}^{B}p_{C}}$$

Simultaneous Rotational & Translational Velocities

Differentiating

$$\mathbf{{}^{A}U_{C} = {}^{A}U_{B} + {}^{A}R_{B}\,{}^{B}U_{C} + \underbrace{{}^{A}\dot{R}_{B}\,{}^{B}p_{C}}}$$

Contribution of rotational velocity of frame B to the translational velocity of C =

$$\begin{aligned}
\mathbf{{}^{A}\dot{R}_{B}\,{}^{B}p_{C}} &= \lfloor \mathbf{{}^{A}w_{B}} \times \rfloor\, \mathbf{{}^{A}R_{B}\,{}^{B}p_{C}} = \mathbf{{}^{A}w_{B}} \times (\mathbf{{}^{A}R_{B}\,{}^{B}p_{C}}) \\
&= -\,\mathbf{{}^{A}R_{B}\,{}^{B}p_{C}} \times \mathbf{{}^{A}w_{B}} \\
&= \mathbf{{}^{A}w_{B}} \times (\mathbf{{}^{A}p_{C}} - \mathbf{{}^{A}p_{B}})
\end{aligned}$$

$$\begin{aligned}
\therefore\ \mathbf{{}^{A}U_{C}} &= \mathbf{{}^{A}U_{B} + {}^{A}R_{B}\,{}^{B}U_{C} + {}^{A}w_{B}} \times (\mathbf{{}^{A}R_{B}\,{}^{B}p_{C}}) \\
&= \mathbf{{}^{A}U_{B} + {}^{A}R_{B}\,{}^{B}U_{C} + {}^{A}w_{B}} \times (\mathbf{{}^{A}p_{C}} - \mathbf{{}^{A}p_{B}})
\end{aligned}$$

Simultaneous Rotational & Translational Velocities

$$ {}^{A}\mathbf{R}_{C} = {}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{R}_{C} $$

$$ {}^{A}\dot{\mathbf{R}}_{C} = {}^{A}\dot{\mathbf{R}}_{B}\,{}^{B}\mathbf{R}_{C} + {}^{A}\mathbf{R}_{B}\,{}^{B}\dot{\mathbf{R}}_{C} $$

$$ {}^{A}\hat{\mathbf{w}}_{C}\;{}^{A}\mathbf{R}_{C} = {}^{A}\hat{\mathbf{w}}_{B}\;{}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{R}_{C} + {}^{A}\mathbf{R}_{B}\;{}^{B}\hat{\mathbf{w}}_{C}\;{}^{B}\mathbf{R}_{C} $$

$$ = {}^{A}\hat{\mathbf{w}}_{B}\,{}^{A}\mathbf{R}_{C} + {}^{A}\mathbf{R}_{B}\,{}^{B}\hat{\mathbf{w}}_{C}\,{}^{B}\mathbf{R}_{A}\,{}^{A}\mathbf{R}_{C} $$

$$ {}^{A}\hat{\mathbf{w}}_{C}\,\cancel{{}^{A}\mathbf{R}_{C}} = {}^{A}\hat{\mathbf{w}}_{B}\,\cancel{{}^{A}\mathbf{R}_{C}} + {}^{A}\mathbf{R}_{B}\;{}^{B}\hat{\mathbf{w}}_{C}\,{}^{B}\mathbf{R}_{A}\,\cancel{{}^{A}\mathbf{R}_{C}} $$

$$ {}^{A}\hat{\mathbf{w}}_{C} = {}^{A}\hat{\mathbf{w}}_{B} + {}^{A}\mathbf{R}_{B}\;{}^{B}\hat{\mathbf{w}}_{C}\,{}^{B}\mathbf{R}_{A} $$

Simultaneous Rotational & Translational Velocities

But

$$ {}^{A}_{B}\hat{w}_{C} = {}^{A}\mathbf{R}_{B}\;{}^{B}\hat{\mathbf{w}}_{C}\,{}^{B}\mathbf{R}_{A} \Leftrightarrow $$

$$ {}^{A}_{B}\mathbf{w}_{C} = {}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{w}_{C} $$ expressing vector in diff frame

$$ \therefore\; {}^{A}\hat{\mathbf{w}}_{C} = {}^{A}\hat{\mathbf{w}}_{B} + {}^{A}_{B}\hat{w}_{C} $$

OR in vector form:

$$ {}^{A}\mathbf{w}_{C} = {}^{A}\mathbf{w}_{B} + {}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{w}_{C} $$

Note also (in homogeneous transformation)

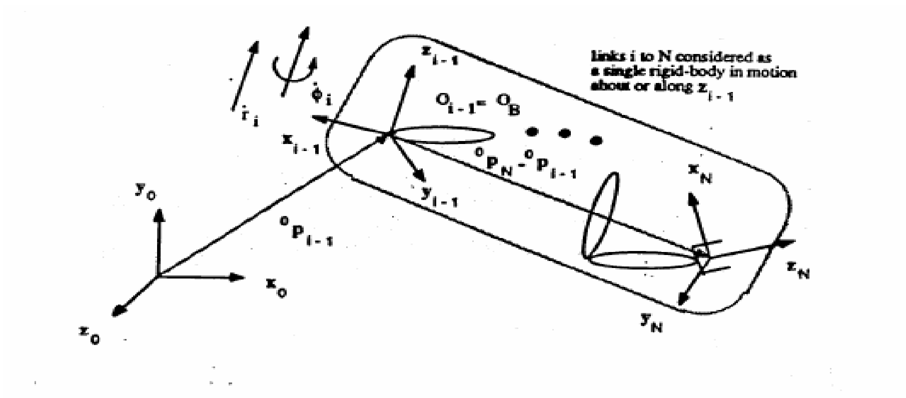
$$ {}^{A}\dot{T}_{B} = \left[\begin{array}{c:c} {}^{A}\hat{w}_{B}\;{}^{A}R_{B} & {}^{A}U_{B} \\ \hdashline 0 & 0 \end{array}\right] $$

Computation Of End-Effector Velocity

(6x1) $\quad \mathbf{v}_N = \begin{pmatrix} u_N \\ w_N \end{pmatrix} = f\,(\,q, \mathring{q}\,)$

joint velocities

joint position



Computation Of End-Effector Velocity

Let us examine the contribution of the ith joint motion to end-effector velocity. We set all other joint velocities ϕ:

$\dot{\mathbf{q}}_C \neq 0 \qquad \dot{\mathbf{q}}_1 = \dot{\mathbf{q}}_2 = \ldots = \dot{\mathbf{q}}_{i-1} = \dot{\mathbf{q}}_{i+1} = \ldots \dot{\mathbf{q}}_N = \phi$

so motion is occurring with respect to $\mathbf{z}_{i-1}$ axis

For joint i rotational

$$\mathbf{w}_i = \mathbf{z}_{i-1}\,\dot{\mathbf{q}}_i$$

$$\mathbf{u}_i = \mathbf{w}_i \times \mathbf{R}_{i-1}\,{}^{i-1}\mathbf{p}_N = \mathbf{z}_{i-1}\,\dot{\mathbf{q}}_i \times (\,\mathbf{p}_N - \mathbf{p}_{i-1}\,)$$

$$= \mathbf{z}_{i-1} \times (\,\mathbf{p}_N - \mathbf{p}_{i-1}\,)\,\dot{\mathbf{q}}_i$$

Note that $\mathbf{o}_{i-1}$ has no translational velocity

origin of frame i-1 w/c contains $\mathbf{z}_{i-1}$ since joint is rotational

Computation Of End-Effector Velocity

For a translational joint i,

$$\mathbf{w_i} = 0$$

$$\mathbf{u_i} = \mathbf{z_{i-1}}\, \dot{\mathbf{q}}_\mathbf{i}$$

The total velocity of the end-effector during coordinated motion is the superposition of all the elementary velocities that represent single joint motion:

$$\mathbf{v_N} = \begin{bmatrix} \mathbf{u_N} \\ \mathbf{w_N} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} u_i \\ \sum_{i=1}^{N} w_i \end{bmatrix}$$

Computation Of End-Effector Velocity

$$\mathbf{v_N} = \underbrace{(\overset{6\text{x}1}{J_1}\ \ J_2\ \ J_3\ \ \ldots\ \ J_N\ )}_{6\text{xN } J(q)} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_N \end{pmatrix}$$

Column J_i represents motion contribution of joint i

J(q) = Jacobian matrix
Cartesian $\leftrightarrow$ joint space

Computation Of End-Effector Velocity

For a translational joint i

$$\mathbf{J_i} = \begin{bmatrix} z_{i-1} \\ 0 \end{bmatrix}$$

For a rotational joint i

$$\mathbf{J_i} = \begin{bmatrix} z_{i-1} \times (p_N - p_{i-1}) \\ z_{i-1} \end{bmatrix}$$

Jacobian Transformations

- Velocities expressed in different frames

$$\mathbf{^A v_N} \leftrightarrow \mathbf{^B v_N} \quad \left\{ \begin{array}{l} \text{N = End Effector} \\ \text{B = may be a link coord} \\ \quad\text{frame that is held} \\ \quad\text{instantaneously constant} \end{array} \right.$$

For $\mathbf{^A R_B}$ and $\mathbf{^A p_B}$ constants

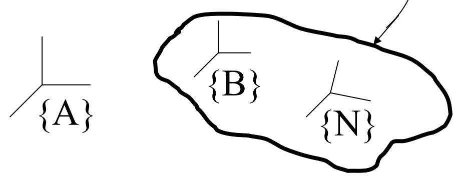
$$^A\mathbf{v}_N = \begin{pmatrix} ^A\mathbf{u}_N \\ ^A\mathbf{w}_N \end{pmatrix} = \underbrace{\left[\begin{array}{c|c} ^A\mathbf{R}_B & 0 \\ \hline 0 & ^A\mathbf{R}_B \end{array}\right]}_{\mathbf{J}} \underbrace{\begin{pmatrix} ^B\mathbf{u}_N \\ ^B\mathbf{w}_N \end{pmatrix}}_{^B\mathbf{v}_N}$$

Jacobian Transformations

- Diff pts on End-Effector

For $\mathbf{{}^{B}R_{N}}$ and $\mathbf{{}^{B}p_{N}}$ constants

B & N are attached to a <u>rigid body</u> moving with respect to A:



N = } Two Frames Attached to End-Effector
B = }

Jacobian Transformations

$$\mathbf{{}^{A}u_{N}} = \mathbf{{}^{A}u_{B}} + \mathbf{{}^{A}w_{B}} \times (\mathbf{{}^{A}p_{N}} - \mathbf{{}^{A}p_{B}})$$

$$\mathbf{{}^{A}w_{N}} = \mathbf{{}^{A}w_{B}}$$

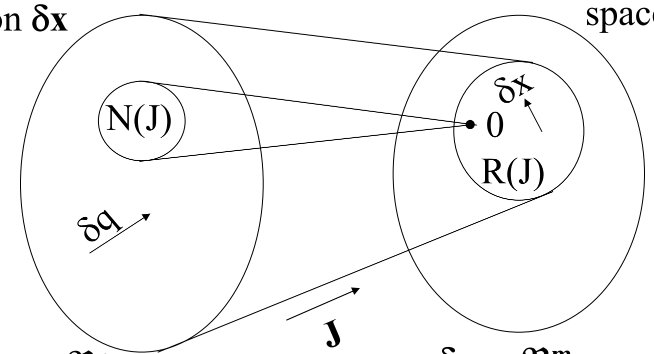
$$ {}^{A}v_{N} = \begin{pmatrix} {}^{A}u_{N} \\ {}^{A}w_{N} \end{pmatrix} = \underbrace{\left[\begin{array}{c:c} I & -({}^{A}p_{N} - {}^{A}p_{B}) \\ \hdashline 0 & I \end{array}\right]}_{\text{another } \mathbf{J}} \begin{bmatrix} {}^{A}u_{B} \\ {}^{A}w_{B} \end{bmatrix} $$

Robot Kinematics of Velocity

N(J) = Null space of J

$\boldsymbol{\delta q} \rightarrow$ produces no motion $\boldsymbol{\delta x}$

R(J) = Range Space of **J** = or column space of **J**



$\boldsymbol{\delta q} \in \Re^n$
Joint Space

$\boldsymbol{\delta x} \in \Re^m$
End-Effector Space

Inverse Kinematics of Velocity

Solution to $\underbrace{\boldsymbol{\delta x}}_{mx1} = \mathbf{J}\, \underbrace{\delta \mathbf{q}}_{nx1}$ [i.e., given $\boldsymbol{\delta x}$, Find $\boldsymbol{\delta q}$]

Exists if & only if

$$\text{Rank } \underbrace{\mathbf{J}}_{m \times n} = \text{Rank } \underbrace{(\ \mathbf{J} \mid \boldsymbol{\delta x}\ )}_{m \times (n+1)}$$

m x (n + 1) matrix obtained by augmenting **J** with column $\boldsymbol{\delta x}$

Meaning $\boldsymbol{\delta x}$ must be in the subspace spanned by the columns of **J**

Inverse Kinematics of Velocity

First: Convert $\delta \mathbf{x}$ to $\delta \mathbf{x_0} \in \mathbf{R}^{\mathbf{m_o}}$(velocity, basic kinematic model)

$$\underbrace{\delta \mathbf{x_0}}_{\mathbf{R}^{\mathbf{m_o}}} = \mathbf{J_0} \underbrace{\mathbf{c}}_{\mathbf{R}^{\mathbf{n}}} \qquad ({}^{\mathbf{0}}\dot{\mathbf{v}}_{\mathbf{N}} = \mathbf{J_0}\, \dot{\mathbf{q}})$$

$$m_0 \leq 6$$

Solution exists if & only if $\underline{\text{Rank } \mathbf{J_0} = \min(m_0, n)}$

i.e., columns of $\mathbf{J_0}$ span the space $\mathbf{R}^{\mathbf{min(\ m_o,n\ )}}$

Inverse Kinematics of Velocity

General Solution:

generalized Inverse

$$\delta \mathbf{q} = \mathbf{J_0}^{\#}(\mathbf{q})\ \delta \mathbf{x_0} + [\ \underbrace{\mathbf{I_n}}_{\text{nxn Identity}} - \mathbf{J_0}^{\#}(\mathbf{q})\ \mathbf{J_0}(\mathbf{q})\ ]\ \underbrace{\delta \mathbf{q_0}}_{\text{any arbitrary disp}}$$

Operates on $\delta \mathbf{q_0}$ to produce vector $\delta \mathbf{q_n} \in \mathbf{N(J)}$

$$\delta \mathbf{q_n} = [\ \mathbf{I_n} - \mathbf{J_0}^{\#}(\mathbf{q})\ \mathbf{J_0}(\mathbf{q})\ ]\ \delta \mathbf{q_0}$$

The mapping by $\mathbf{J_0}$ of $\delta \mathbf{q_n}$ results in zero vector in $\mathbf{R}^{\mathbf{m_o}}$

$$\mathbf{J_0}\, \delta \mathbf{q_n} = [\mathbf{J_0} - \mathbf{J_0}\, \mathbf{J_0}^{\#}(\mathbf{q})\ \mathbf{J_0}(\mathbf{q})\ ]\ \delta \mathbf{q_0} = 0$$

Generalized Inverse

- The generalized inverse of A (*m x n*) is A* (*n x m*) , such that

$$AA^*A = A$$

Example:

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A^* = \begin{pmatrix} 2 & -3 & a \\ -1 & 2 & b \\ c & d & e \end{pmatrix}$$

a, b, c, d, e can be any number. If they are all zero, this is pseudo inverse (or Moore-Penrose Generalized Inverse)

- Note that

$$A^*A \neq I \qquad\qquad AA^* \neq I$$

In previous example,

$$A^*A = AA^* = \begin{pmatrix} 4 & -9 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Inverse Kinematics of Velocity

Case 1: $m_0 = n \quad \leq 6$

$\mathbf{J^{\#}(q) = J^{-1}}$ (possible problem with singularity, $\mathbf{J^{-1}}$ may not exist)

Case 2: $m_0 > n, \quad m_0 \leq 6$ (not interesting/useful case, task shall be $\leq n$)

overdetermined system: more eqns than unknowns.

$\mathbf{J^{\#} = (J^T J^{-1}) J^T}$ = left pseudo inverse

= exists only if Rank $\mathbf{J} = n$

Sol'n minimizes $\| \mathbf{J\delta q - \delta x_0} \|_2$

Inverse Kinematics of Velocity

Case 3: $m_0 < n, \quad m_0 \leq 6$ (Redundant Robots)

underdetermined system = less eqns than unknowns

$\mathbf{J^{\#} = J^T (J J^T)^{-1}}$ = right pseudo inverse

= exists only if Rank $\mathbf{J} = m_0$

Sol'n minimizes $\| \mathbf{\delta q} \|_2$

Resolved Motion Rate Control

– Kinematic Control without the need for solving the inverse Kinematics of Position

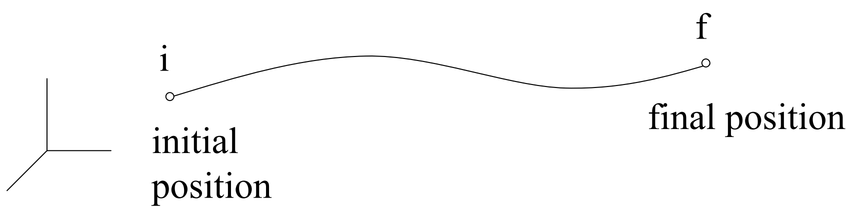
– Need Joint level control which is available in robot controllers

δq δx

– Command joint motion such that desired e-e motion is achieved

Resolved Motion Rate Control

1) Given a Trajectory $x(t) \in R^m$ in task space



2) Divide Trajectory into small segments according to sample time on reference Trajectory update rate

3) At x_k, compute $J(q_k)$

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4) Compute $\Delta x_k = x_{k+1} - x_k$ (position and orientation error)

5) Convert orientation error into 3 x 1 vector ($d\Phi$)

6) Compute $\Delta q = J^{\#}(q_k)\ \Delta\ x_{0,\,k} + [In - J_0^{\#}(q^k)\ J_0(q_k)]\ \Delta q_0$

7) Command δq to robot controller
   (Robot moves from q_k to q_{k+1}) ($\delta q = q_{k+1} - q_k$)

8) Go to step 3 until x_k reaches x_f