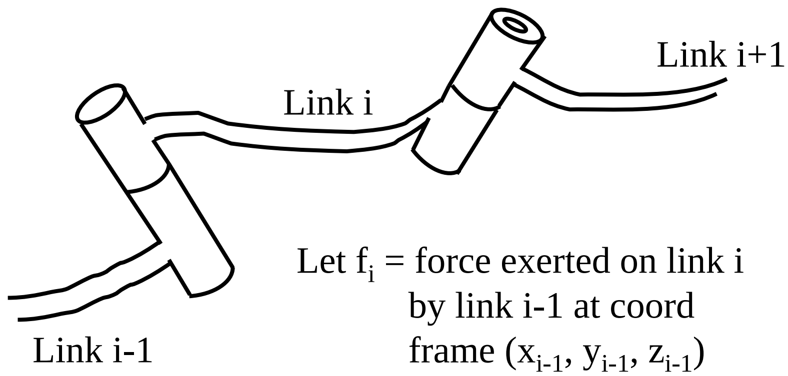
CHAPTER 4

Force Transformation and Robot Statics

Learning Objectives

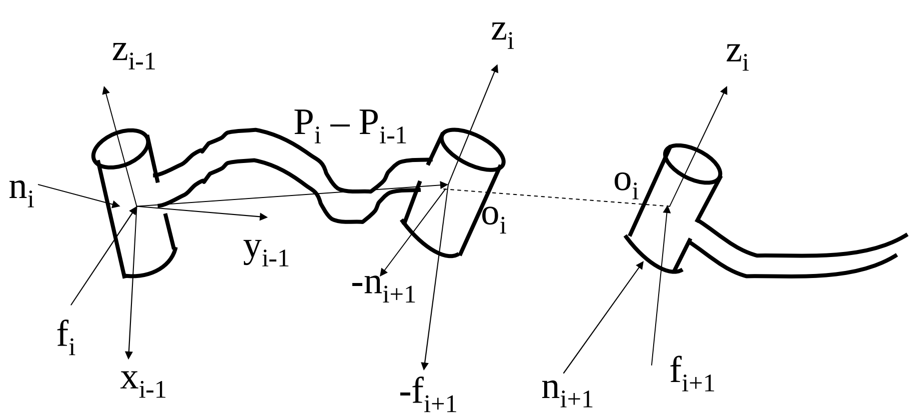
- Relate joint actuator forces with end-effector forces
- Transform forces in different frames

Static Forces in Manipulators



Let f_i = force exerted on link i
by link i-1 at coord
frame $(x_{i-1}, y_{i-1}, z_{i-1})$
n_i = moment exerted on link i

Static Forces in Manipulators



Static Forces in Manipulators

$$\begin{cases} \sum \mathbf{F} = 0 \quad \mathbf{f_i} - \mathbf{f_{i+1}} = 0 \\ \sum \text{Torques about origin of frame i-1} = 0 \\ \quad \mathbf{n_i} - \mathbf{n_{i+1}} + (\mathbf{p_i} - \mathbf{p_{i-1}}) \times (-\mathbf{f_{i+1}}) = 0 \end{cases}$$

If we start with a description of the force and moment applied by the hand, we can calculate the force and moment applied by each link working from the last link down to the base, link ϕ.

$\left.\begin{matrix} \mathbf{f_{n+1}} \\ \mathbf{n_{n+1}} \end{matrix}\right\}$ Force exerted by the manipulator hand on its environment.

Static Forces in Manipulators

Recursive Equations:

$$\left.\begin{matrix} \mathbf{f_i} = \mathbf{f_{i+1}} \\ \mathbf{n_i} = \mathbf{n_{i+1}} + (\mathbf{p_i} - \mathbf{p_{i-1}}) \times \mathbf{f_{i+1}} \end{matrix}\right\}$$ all vectors expressed in same frame (e.g. base frame ϕ)

What forces are Needed at the Joints in order to Balance the Reaction Forces & Moments acting in the link

$$\mathbf{T_i} = \begin{cases} \mathbf{n_i^T\, z_{i-1}} & \text{for a rotational link i} \\ \mathbf{f_i^T\, z_{i-1}} & \text{for a translational link i} \end{cases}$$

Jacobians In Force Domain

- When forces act on a mechanism, work (in the technical sense) is done if the mechanism moves through a displacement

- Principle if VIRTUAL WORK allows us to make certain statements about the static case by defining a VIRTUAL DISPLACEMENT δx that is experienced without passage of time $dt = 0$
(Not only infinitesimal, $dx \neq \delta x$)

Jacobians In Force Domain

- Since work has units of energy, it must be the same measured in any set of generalized coordinates

6x1 virtual joint disp.

$$\mathbf{F} \cdot \delta\mathbf{x} = \tau \cdot \delta\mathbf{q}$$

6x1 Cartesian Force-Moment Vector

6x1 Virtual displ. In Cartesian space

Torque/Forces at joints 6x1

Jacobians In Force Domain

- But $\delta\mathbf{x} = \mathbf{J}\,\delta\mathbf{q}$

- Therefore $\mathbf{F}^{\mathbf{T}}\,\underbrace{[\mathbf{J}\,\delta\mathbf{q}]}_{\delta\mathbf{x}} = \underline{\tau^{\mathbf{T}}\,\delta\mathbf{q}}$

$$\mathbf{F}^{\mathbf{T}}\,\mathbf{J} = \tau^{\mathbf{T}}$$

$$\boxed{\tau = \mathbf{J}^{\mathbf{T}}\,\mathbf{F}}$$

expressed in the same (<u>consistent</u>) Frame

Valid only at non-singular configurations

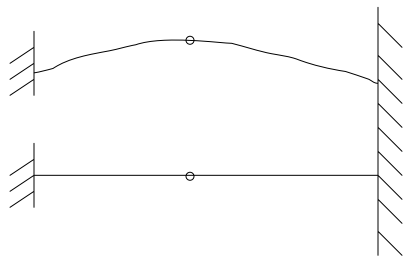
Jacobians In Force Domain

- When the Jacobian loses full rank, there are certain directions in which the end-effector cannot exert static forces (through joint actuation) as desired

- That is, if **J** is singular, the equation is not valid
  - **F** could be increased or decreased in certain directions with no effect on the value calculated for τ
  - These directions are in the null-space of the Jacobian

Jacobians In Force Domain

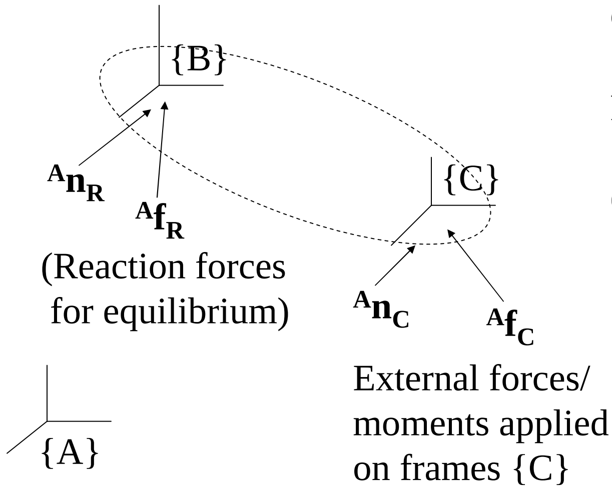
- This also means that near singular configuration, mechanical advantage tends towards infinity, such that with small joint torques, large forces could be generated at the end-effector



Jacobians In Force Domain

- Note that a Cartesian space quantity can be converted into a joint space quantity without calculating any inverse kinematic functions.

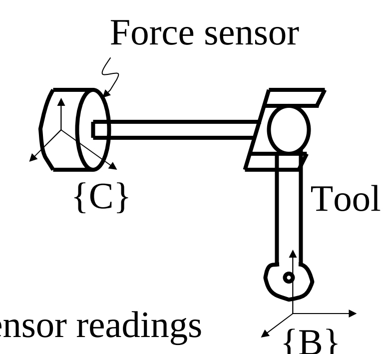
Cartesian Transformation Of Static Force



Given: ${}^A f_C$
${}^A n_C$
Find: ${}^A f_B$
${}^A n_B$
(the force/moment experienced at B if force/moment is exerted on C)

Cartesian Transformation Of Static Force

Why is this important?



${}^C\mathbf{f}_C$ & ${}^C\mathbf{n}_C$ can be force sensor readings
But our primary interest is ${}^B\mathbf{f}_B$ & ${}^B\mathbf{n}_B$
(force/moments at tool tip)

Cartesian Transformation Of Static Force

Equilibrium:

$\sum F = 0$ $\quad$ ${}^{A}f_{C} + {}^{A}f_{R} = 0$

$\quad\quad\quad\quad$ ${}^{A}f_{R} = - {}^{A}f_{C}$

$\sum N = 0$ $\quad$ ${}^{A}n_{C} + ({}^{A}p_{B} - {}^{A}p_{C}) \times {}^{A}f_{R} + n_{R} = 0$

$\quad\quad\quad\quad$ ${}^{A}n_{R} = - {}^{A}n_{C} - ({}^{A}p_{B} - {}^{A}p_{C}) \times {}^{A}f_{R}$

But ${}^{A}f_{B} = - {}^{A}f_{R}$ $\rightarrow$

$$\boxed{{}^{A}f_{B} = {}^{A}f_{C}}$$

Cartesian Transformation Of Static Force

$${}^{A}\mathbf{n}_{B} = -{}^{A}\mathbf{n}_{R} = {}^{A}\mathbf{n}_{C} + ({}^{A}\mathbf{p}_{B} - {}^{A}\mathbf{p}_{C}) \times {}^{A}\mathbf{f}_{R}$$

$${}^{A}\mathbf{n}_{B} = {}^{A}\mathbf{n}_{C} + ({}^{A}\mathbf{p}_{B} - {}^{A}\mathbf{p}_{C}) \times (-{}^{A}\mathbf{f}_{C})$$

$$\boxed{{}^{A}\mathbf{n}_{B} = {}^{A}\mathbf{n}_{C} + ({}^{A}\mathbf{p}_{C} - {}^{A}\mathbf{p}_{B}) \times {}^{A}\mathbf{f}_{C}}$$

OR

$${}^{A}\mathbf{n}_{B} = {}^{A}\mathbf{n}_{C} + \lfloor {}^{A}\mathbf{R}_{B}\, {}^{B}\mathbf{p}_{C} \mathbf{x} \rfloor\; {}^{A}\mathbf{f}_{C}$$

in Matrix Form

$$\begin{bmatrix} {}^{A}\mathbf{f}_{B} \\ {}^{A}\mathbf{n}_{B} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{I} & 0 \\ \lfloor {}^{A}\mathbf{R}_{B}\, {}^{B}\mathbf{p}_{C} \mathbf{x} \rfloor & \mathbf{I} \end{bmatrix}}_{\text{Force torque Jacobian transformation}} \begin{bmatrix} {}^{A}\mathbf{f}_{C} \\ {}^{A}\mathbf{n}_{C} \end{bmatrix}$$

Cartesian Transformation Of Static Force

But in typical applications, we would like to relate

$$\begin{bmatrix} {}^{C}\mathbf{f}_{C} \\ {}^{C}\mathbf{n}_{C} \end{bmatrix} \text{with} \begin{bmatrix} {}^{B}\mathbf{f}_{B} \\ {}^{B}\mathbf{n}_{B} \end{bmatrix}$$

[e.g. sensor readings will be expressed in local frame of sensor]

We can transform vectors **f** & **n** like any other vector via Rotation Matrices

$$\begin{bmatrix} {}^{A}\mathbf{f}_{C} \\ {}^{A}\mathbf{n}_{C} \end{bmatrix} = \left[\begin{array}{c:c} {}^{A}\mathbf{R}_{C} & 0 \\ \hdashline 0 & {}^{A}\mathbf{R}_{C} \end{array}\right] \begin{bmatrix} {}^{C}\mathbf{f}_{C} \\ {}^{C}\mathbf{n}_{C} \end{bmatrix}$$

Cartesian Transformation Of Static Force

$\therefore$

$$\begin{bmatrix} {}^{A}\mathbf{f}_{B} \\ {}^{A}\mathbf{n}_{B} \end{bmatrix} = \left[\begin{array}{c:c} \mathbf{I} & 0 \\ \hdashline \lfloor {}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{p}_{C}\mathbf{x} \rfloor & \mathbf{I} \end{array}\right] \left[\begin{array}{c:c} {}^{A}\mathbf{R}_{C} & 0 \\ \hdashline 0 & {}^{A}\mathbf{R}_{C} \end{array}\right] \begin{bmatrix} {}^{C}\mathbf{f}_{C} \\ {}^{C}\mathbf{n}_{C} \end{bmatrix}$$

$$\begin{bmatrix} {}^{A}\mathbf{f}_{B} \\ {}^{A}\mathbf{n}_{B} \end{bmatrix} = \left[\begin{array}{c:c} {}^{A}\mathbf{R}_{C} & 0 \\ \hdashline \lfloor {}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{p}_{C}\mathbf{x} \rfloor\,{}^{A}\mathbf{R}_{C} & {}^{A}\mathbf{R}_{C} \end{array}\right] \begin{bmatrix} {}^{C}\mathbf{f}_{C} \\ {}^{C}\mathbf{n}_{C} \end{bmatrix}$$

Also

$$\begin{bmatrix} {}^{B}\mathbf{f}_{B} \\ {}^{B}\mathbf{n}_{B} \end{bmatrix} = \left[\begin{array}{c:c} {}^{B}\mathbf{R}_{A} & 0 \\ \hdashline 0 & {}^{B}\mathbf{R}_{A} \end{array}\right] \begin{bmatrix} {}^{A}\mathbf{f}_{B} \\ {}^{A}\mathbf{n}_{B} \end{bmatrix}$$

Cartesian Transformation Of Static Force

Therefore

$$\begin{bmatrix} {}^B f_B \\ {}^B n_B \end{bmatrix} = \left[\begin{array}{c|c} {}^B R_A & 0 \\ \hline 0 & {}^B R_A \end{array}\right] \left[\begin{array}{c|c} {}^A R_C & 0 \\ \hline \widehat{\left({}^A R_B\, {}^B p_C\right)}\, {}^A R_C & {}^A R_C \end{array}\right] \begin{bmatrix} {}^C f_C \\ {}^C n_C \end{bmatrix}$$

$$= \left[\begin{array}{c|c} {}^B R_A\, {}^A R_C & 0 \\ \hline \underbrace{{}^B R_A\, \widehat{\left({}^A R_B\, {}^B p_C\right)}\, {}^A R_C} & {}^B R_A\, {}^A R_C \end{array}\right] \begin{bmatrix} {}^C f_C \\ {}^C n_C \end{bmatrix}$$

$(\mathbf{{}^B R_A}\, \widehat{(\mathbf{{}^A R_B}\, \mathbf{{}^B p_C})}\, \mathbf{{}^A R_C})\, \mathbf{{}^C f_C} = \mathbf{{}^B R_A}\, [(\mathbf{{}^A R_B}\, \mathbf{{}^B p_C}) \times (\mathbf{{}^A R_C}\, \mathbf{{}^C f_C})]$

$= (\mathbf{{}^B R_A}\, \mathbf{{}^A R_B}\, \mathbf{{}^B p_C}) \times (\mathbf{{}^B R_A}\, \mathbf{{}^A R_C}\, \mathbf{{}^C f_C})$

$= \mathbf{{}^B p_C} \times (\mathbf{{}^B R_C}\, \mathbf{{}^C f_C})$

$= \widehat{\mathbf{{}^B p_C}}\, \mathbf{{}^B R_C}\, \mathbf{{}^C f_C}$

Cartesian Transformation Of Static Force

$$\begin{bmatrix} {}^B f_B \\ {}^B n_B \end{bmatrix} = \left[\begin{array}{c|c} {}^B R_C & 0 \\ \hline \lfloor \widehat{{}^B p_C} \rfloor\, {}^B R_C & {}^B R_C \end{array}\right] \begin{bmatrix} {}^C f_C \\ {}^C n_C \end{bmatrix}$$

This is the form given in Craig's Book